Robust Mode Space Supergain Beamforming under Unknown Array Mismatch

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Abstract—**The pure supergain beamformers, or MVDR beamformers have better resolution and array gain than conventional delay-and-sum beamformers (CBF), yet are highly sensitive to errors in array parameters. In this paper a robust supergain beamforming (RSBF) method is proposed for circular arrays under unknown array mismatch. The robustness of the array is considered as an optimization objective instead of constraint and it is observed the proposed robust supergain beamforming belongs to the class of diagonal loading approaches. The amount of diagonal loading can be optimally selected as a function of frequency under unknown array mismatch. A closed-form optimal solution is thus obtained for robust supergain beamforming under unknown array mismatch. Simulation results show the performance of the robust supergain beamforming surpasses the conventional beamformer and MVDR beamformer under elevated noise levels, especially at low frequencies.**

Keywords-robust beamforming, circular array, supergain, diagonal loading, optimization

I. INTRODUCTION

Beamforming is an important stage of sensor array signal processing and its applications range from radar, sonar, acoustics, astronomy, communications, and medical imaging. Circular arrays are widely used in underwater acoustics array signal processing due to their two dimensional symmetrical characteristics, i.e., the ability to form uniform beams along their azimuthal directions [1]. Generally there are two kinds of beamforming techniques for circular arrays. The first one is based on the delay-and-sum of the weighted sensor data, which is referred to as sensor space approach. The second technique is based on the concept of phase modes and is generally referred to as mode space approach [2-4]. MVDR beamformer has better resolution and much higher array gain than the conventional delay-and-sum beamformer, provided that the array sensors are accurately positioned and manufactured as described [1, 3-4]. However, array mismatch often occurs in practice due to the differences between the assumed array sensor position and true sensor position, or between the manufactured array sensors' nominal phase / gain and the true sensors' phase / gain. In these cases, the performance of the MVDR beamformer may become worse than that of the conventional beamformers.

To improve the robustness of the supergain beamformer, White Noise Gain (WNG) constraint, as a measurement of the array's robustness performance, is usually imposed. Diagonal loading has been a popular approach to improve the robustness of the MVDR beamformer [1, 5-6]. However, the main problem associated with diagonal loading method is that it is not clear how to choose the diagonal loading factor. Cox [7] proposed to use the WNG constraint to obtain reasonable values of diagonal loading factor. Unfortunately, the relationship between the WNG constraint and diagonal loading factor is not simple and a multistep iterative procedure is required to obtain the diagonal loading factor, which may be computationally inefficient.

Researchers have also proposed robust supergain beamformers based on convex optimization method, both in sensor space [8,9] and model space [10-13]. These methods can make good trade-off among Array Gain (AG) and WNG, but the value of the WNG constraint is generally hard to find and the relationship between the WNG constraint and the degree of array mismatch is not clear. Specifically, Yong [12] studied a direct robust supergain method in mode space for circular arrays mounted on an infinite rigid cylinder. Second-order cone programing (SOCP) is utilized to obtain the robust optimal mode weighting vector under various constraints including WNG and sidelobe level. The authors suggested to use low order modes in low frequencies and increase the mode order as the frequency becomes higher. However, the criteria of changing the mode order under different frequencies remains unsolved. Moreover, in practical applications the array mismatch is usually unknown. In this case the WNG may not be considered as a fixed value of constraints, but rather a measurement that should be optimized together with the AG.

In this paper, the robustness of the array is considered as an optimization objective rather than a constraint and it is observed that the proposed robust supergain beamforming belongs to the class of diagonal loading approaches. Moreover, the amount of diagonal loading can be optimally selected as a function of frequency under unknown array mismatch. The proposed optimal supergain beamforming robust to unknown array mismatch is verified by computer simulation. Results suggest that MVDR beamformer fails when large array mismatch occurs. The proposed beamformer, however, is robust against large array mismatch, while processes a

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relatively larger array gain and narrower mainlobe bandwidth compared to the conventional beamformer, especially at low frequencies.

II. Beamforming in Mode Space

Consider a circular array with M sensors. A plane wave with unity magnitude impinging from (θ,ϕ), and the sound pressure received by the array element located at $(\theta_m,\phi_m,0)$, where r_m=a, is expressed as follows [2]:

$$p_m(\theta,\phi,ka)=e^{ika\sin\theta\cos(\phi_m-\phi)}=\sum_{n=-\infty}^{\infty}b_n(ka,kr_m,\theta)e^{in(\phi_m-\phi)} \quad (1)$$

where a is the sphere radius, k is the wave number, (r_m,θ_m,ϕ_m) is the location of the array elements, and i denotes complex unit. To simplify notation, $b_n(ka,kr_m,\theta)$ is introduced both for Rigid Cylinder (RC) and Open Cylinder (OC) [12]:

$$b_n(ka,kr_m,\theta)=\begin{cases} i^n\left[J_n(kr_m\sin\theta)-\dfrac{J_n^{'}(ka\sin\theta)}{H_n^{'}(ka\sin\theta)}H_n(kr_m\sin\theta)\right], & \text{RC}\\ i^nJ_n(kr_m\sin\theta), & \text{OC}\end{cases} \quad (2)$$

where J_n is n*th* order Bessel function and H_n is n*th* order Hankel function. $J_n^{'}$ and $H_n^{'}$ represent their first derivative, respectively. The manifold vector is:

$$P(ka,\theta,\phi)=[p_1(ka,\theta,\phi),\ldots,p_m(ka,\theta,\phi),\ldots,p_M(ka,\theta,\phi)]^T \quad (3)$$

Because the flexibility of the circular arrays in beam steering along the azimuthal directions, this paper focuses on the performance of the azimuthal beam, i.e., the plane wave impinges from $(\pi/2,\phi)$.

The beamformer in mode space consists of two parts: eigenbeamformer and modal-beamformer. The eigenbeamformer performs a spatial decomposition of the sound field into orthogonal circular or spherical harmonics, which are denoted as eigen-beams [12]. For continuous circular aperture, the eigen-beam is expressed as:

$$E_n(ka,kr_m,\phi)=\frac{1}{2\pi}\int_0^{2\pi}p(ka,\phi)e^{-in\phi_m}d\phi_m=b_n(ka,kr_m)e^{-in\phi} \quad (4)$$

The following orthogonal relationship is then satisfied:

$$\frac{1}{2\pi}\int_0^{2\pi}e^{i(n-n')\phi_m}d\phi_m=\delta_{n-n'} \quad (5)$$

where $\delta_{n-n'}$ is Kronecker delta function.

In practical applications the continuous circular aperture is usually implemented by M discrete circular array omnidirectional sensors. The similar expression for the eigen-beam can be expressed as [12]:

$$E_n(ka,kr_m,\phi)=\frac{1}{M}\sum_{m=1}^{M}p(ka,kr_m,\phi)e^{-in\phi_m}\approx b_n(ka,kr_m)e^{-in\phi} \quad (6)$$

The highest order that can be extracted from M sensors is constrained to $N_{\max}\approx k_{\max}a$ .The sampling theorem should also be satisfied to prevent sampling error: $M>2N_{\max}$.

The modal-beamformer forms the output by combining the weighted eigen-beams decomposed by the eigenbeamformer. The output of the modal-beamformer is the beampattern:

$$B(ka,\phi)=\mathbf{a}^H\mathbf{E} \quad (7)$$

where a is the modal weight vector, and $\mathbf{E}=[E_{-N},\ldots E_N]^T$ is the modal manifold vector.

III. Robust Supergain Beamforming

A. Calculating AG and WNG in mode space

Array Gain is defined as the output Signal to Noise Ratio (SNR) of the array divided by the input SNR of the individual array elements. For plane wave propagates in omnidirectional noise field, AG is equivalent to the array's Directivity Factor (DF), which in mode space can be expressed as:

$$AG=DF=\frac{\left|\mathbf{a}^H\mathbf{E}(\theta_0,\phi_0)\right|^2}{\mathbf{a}^H\mathbf{R}_{noise}\mathbf{a}} \quad (8)$$

where $\mathbf{R}_{noise}$ is the spatial unified noise covariance matrix in mode space [12]:

$$\mathbf{R}_{noise}=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\mathbf{E}(\theta,\phi)\mathbf{E}^H(\theta,\phi)\sin\theta d\theta d\phi \quad (9)$$

WNG is the AG of the array when the noise is spatial uncorrelated white noise. Because the phase error and the position error of the array are uncorrelated between each elements, their influences over the performances of the beamformer is similar to those made by the spatial white noise, or WNG. The larger WNG is, the more robust the beamformer is to sensor errors. In mode space, WNG can be expressed as [12]:

$$WNG=\frac{\left|\mathbf{a}^H\mathbf{E}(\theta_0,\phi_0)\right|}{\mathbf{a}^H\mathbf{a}}=\frac{M}{\mathbf{a}^H\mathbf{a}} \quad (10)$$

B. MVDR and CBF in mode space

Theoretically, MVDR beamformer can achieve the largest AG while assuring no distortion in the pointing direction. In mode space, MVDR beamformer can be achieved by optimizing the following equation:

$$\min \quad \mathbf{a}^H\mathbf{R}_{noise}\mathbf{a} \quad s.t. \quad \mathbf{a}^H\mathbf{E}(\theta_0,\phi_0)=1 \quad (11)$$

By applying the langrange method, MVDR beamformer in mode space is expressed as:

$$\mathbf{a}_{MVDR}=\frac{\mathbf{R}_{noise}^{-1}\mathbf{E}(\theta_0,\phi_0)}{\mathbf{E}^H(\theta_0,\phi_0)\mathbf{R}_{noise}^{-1}\mathbf{E}(\theta_0,\phi_0)} \quad (12)$$

$$AG_{MVDR}=\mathbf{E}^H(\theta_0,\phi_0)\mathbf{R}_{noise}^{-1}\mathbf{E}(\theta_0,\phi_0) \quad (13)$$

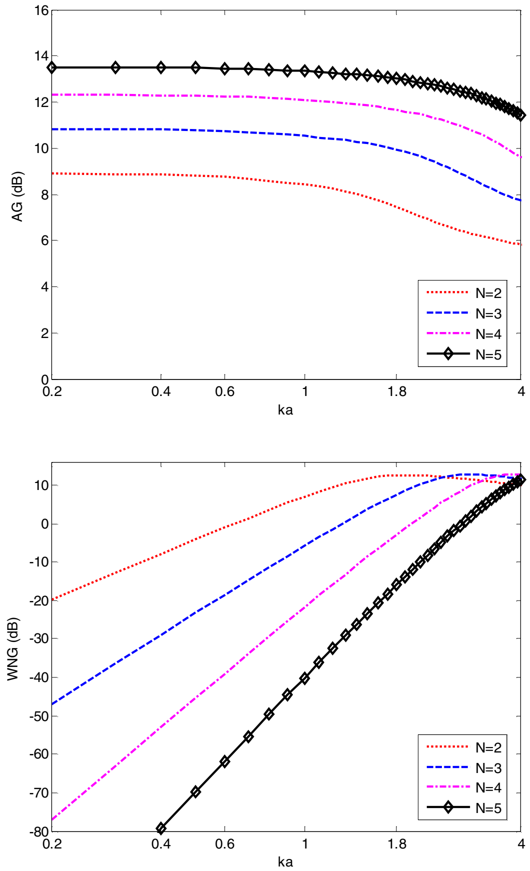


Fig.1 Performance of mode space MVDR beamformer for 12 elements circular array mounted on infinite rigid cylinder. Above: AG; below: WNG

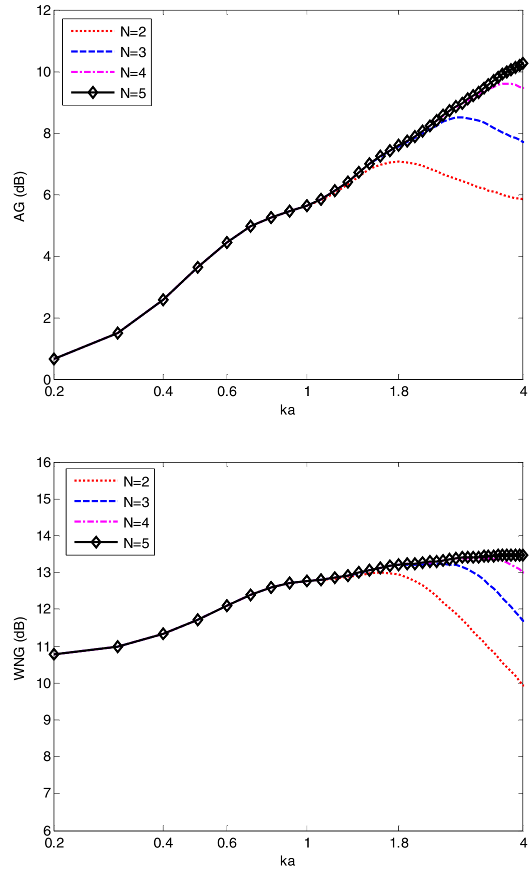


Fig.2 Performance of mode space CBF beamformer for 12 elements circular array mounted on infinite rigid cylinder Above: AG; below: WNG

Similarly, CBF weighting vector in mode space is defined as [12]:

$$\mathbf{a}_{CBF} = \left[\mathbf{E}^H(\theta_0,\phi_0)\mathbf{E}(\theta_0,\phi_0)\right]^{-1}\mathbf{E}(\theta_0,\phi_0) \tag{14}$$

$$AG_{CBF} = \frac{\left[\mathbf{E}^H(\theta_0,\phi_0)\mathbf{E}(\theta_0,\phi_0)\right]^2}{\mathbf{E}^H(\theta_0,\phi_0)\mathbf{R}_{noise}^{-1}\mathbf{E}(\theta_0,\phi_0)} \tag{15}$$

Fig.1 depicts the AG and WNG of mode space MVDR beamformer for circular array with 12 sensors mounted on infinite rigid cylinder. It is observed that for the maximum mode order equals 2, AG remains about 9dB for low frequencies and starts to decrease as the frequency becomes larger. As the maximum mode order extracted from the array increases, AG increases. For the maximum mode order N=3, 4, and 5, the AG is about 10.9dB, 12.2dB, and 13.8dB at low frequencies, respectively. However, as the increase of the mode order brings larger AG, WNG degrades significantly. For N=2, WNG is relatively small (about -20dB) for ka=0.2, indicating the strong sensitivity to small array perturbations. The WNG starts to increase as the frequency increases. For N=3, 4, and 5, the WNG for ka=0.2 equals -48dB, -78dB, and -100dB, respectively. In these cases, mode space MVDR is extremely sensitive to the array mismatch, which makes high order mode space MVDR beamformer impractical to implement.

In Fig.2 the performance of the CBF is illustrated for 12 elements circular array mounted on infinite rigid cylinder. As is seen in Fig.2 (a), the AG for the mode space CBF is very small (about 0.8dB) for ka=0.2, for N of up to 5. As the frequency increases, the AG starts to increase. WNG, on the other hand, remains large for low frequencies (about 10.9dB for ka=0.2), indicating a good robustness for array mismatch. As the frequency increases, the WNG increases (about 13dB for ka=1). Different maximum mode order plays no effect when the frequencies are small. This makes sense because the CBF is not the optimal solution in the sense of AG, and the higher mode is not sufficiently used, especially at low frequencies. However, MVDR beamformer treats high order modes and low order modes equally important for optimization, thus losing its robustness considerably at low frequencies because it tries to amplify the high order mode which is very sensitive to small array errors. In the next subsection, a robustness beamformer appoach is proposed, which can combine the advantages of CBF and MVDR and finds a optimal tradeoff between the AG and WNG.

C. *Robust Supergain Beamforming (RSBF)*

As has been discussed before, WNG is usually considered as a constraint in traditional robust supergain beamforming techniques. In this paper, WNG is considered as a optimization objective. The following optimization objective is proposed:

$$\min\ \mathbf{a}^H\mathbf{R}_{noise}\mathbf{a}+\lambda\mathbf{a}^H\mathbf{a},\quad s.t.\quad \mathbf{a}^H\mathbf{E}(\theta_0,\phi_0)=1 \tag{16}$$

where λ is the optimization coefficient, which correlates the AG and WNG.

The above objective function can be further written as:

$$\min\ \mathbf{a}^H\left(\mathbf{R}_{noise}+\lambda I\right)\mathbf{a},\quad s.t.\quad \mathbf{a}^H\mathbf{E}(\theta_0,\phi_0)=1 \tag{17}$$

Similar to the solution of MVDR beamformer, the following solution is achieved for the proposed RSBF:

$$\mathbf{a}_{RSBF}=\frac{\left(\mathbf{R}_{noise}+\lambda I\right)^{-1}\mathbf{E}(\theta_0,\phi_0)}{\mathbf{E}^H(\theta_0,\phi_0)\left(\mathbf{R}_{noise}+\lambda I\right)^{-1}\mathbf{E}(\theta_0,\phi_0)} \tag{18}$$

$$AG_{RSBF}=\mathbf{E}^H(\theta_0,\phi_0)\left(\mathbf{R}_{noise}+\lambda I\right)^{-1}\mathbf{E}(\theta_0,\phi_0) \tag{19}$$

As can be observed from equation (18), the proposed RSBF weighting vector is in the form of diagonal loading, and the optimization coefficient which correlates the AG and WNG is similar to the diagonal loading amount in the diagonal loading approaches. The difference is that the diagonal loading approach treats WNG as a constraint, and thus it is difficult to obtain the correlation between diagonal loading amount and the pre-set WNG constraint value, especially for the unknown level of array mismatch. While for the proposed RSBF method, the optimization coefficient λ is an coefficient that defines the importance of the two considered optimization objectives, AG and WNG. For the proposed RSBF method, the optimization coefficient λ is crucial in obtaining a good trade-off between the performance and robustness. Fig.3 depicts the AG versus WNG for different *ka*, ranging from 0.2 to 4.0. For each line the optimization coefficient ranges from 0.001 to 10. As can be observed, for *ka*=0.2, as λ increases, the AG decreases at a relatively slow slope, i.e., a small sacrifice of the AG will bring large increase in the WNG.

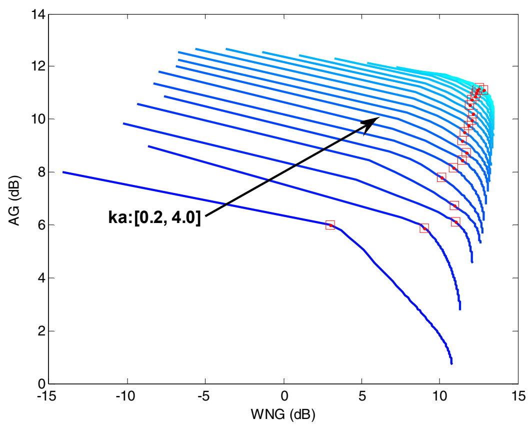


Fig.3 AG versus WNG for the proposed RSBF method. *ka* increases from 0.2 to 4. The red squares indicate the optimal "point" for individual *ka*.

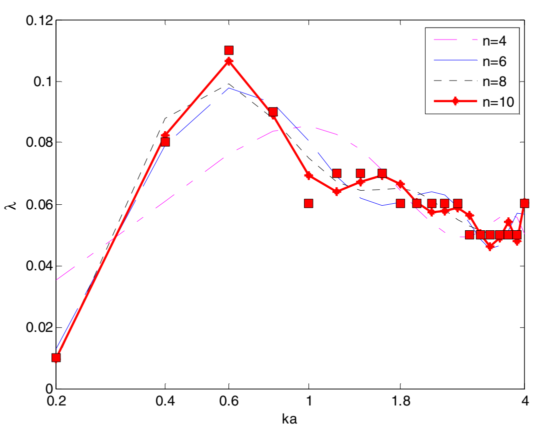


Fig.4 Polynomial fitting of the optimization coefficient over *ka*.

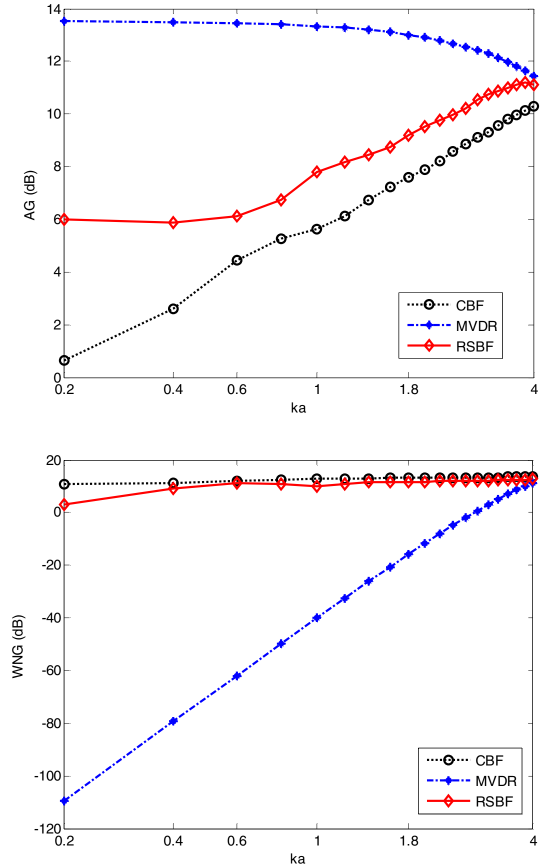
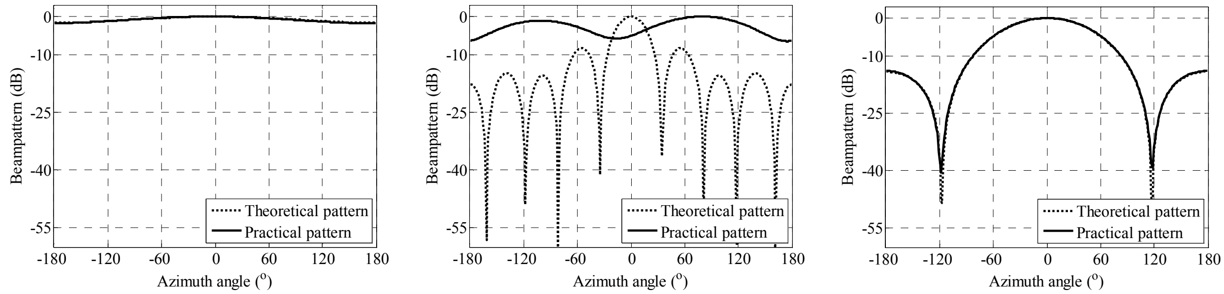


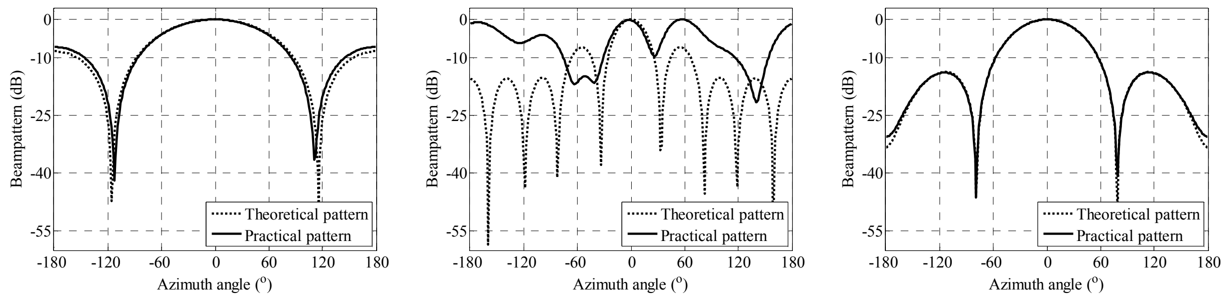
Fig.5 Performance of mode space RSBF beamformer in comparison to the CBF and MVDR beamformers of mode order N=5 for 12 elements circular array mounted on infinite rigid cylinder Above: AG; below: WNG

Specifically, as the AG decreases from 8dB to 6.2dB, the WNG increases from -14dB to 3.5dB. On the other hand, 1.8dB loss in array gain gives 17dB increase in the array robustness measurement. In this particular point (i.e., AG=6.2dB, WNG=3.5dB), the optimization coefficient λ is 0.011. As the coefficient becomes larger, the WNG continues to grow, yet the slope of the decrease in AG becomes larger. It is noted that the determination of the optimal point (whether the decrease of the AG is "worthwhile" for the increase of the WNG) varies from application to application. However, for different application the procedure will remains the same and similar solutions can be achieved. In our study, the determination slope is set to 0.3, which is considered a good tradeoff between the importance of AG and WNG. Thus, for *ka*=0.2, a optimal optimization coefficient (or diagonal loading amount) is chosen as 0.011. By applying similar procedure,



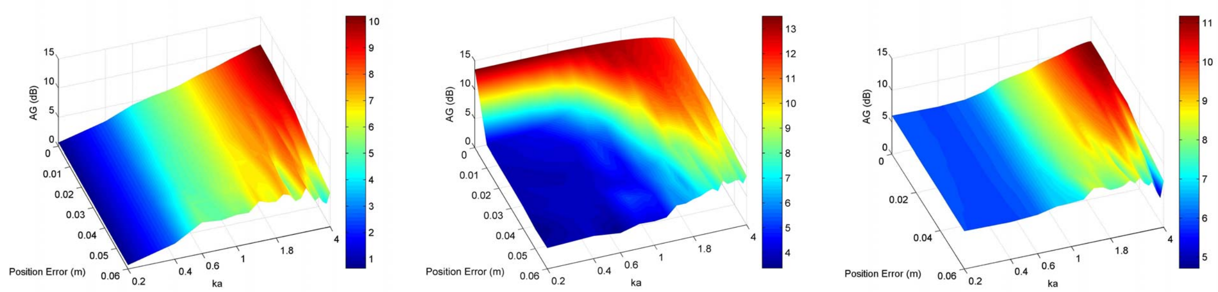
(a) (b) (c)

Fig.6 Theoretical and practical beam pattern of different mode space beamforming technique for *ka*=0.2 (a) CBF (b) MVDR (c) RSBF



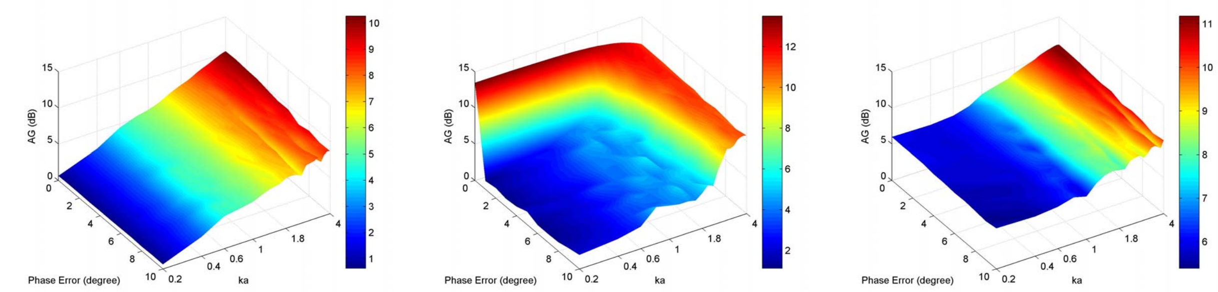
(a) (b) (c)

Fig.7 Theoretical and practical beam pattern of different mode space beamforming technique for *ka*=1.0 (a) CBF (b) MVDR (c) RSBF



(a) (b) (c)

Fig.8 Actual AG under position errors which has Gaussian distributions with variance from 0m to 0.06m for different *ka* (a) CBF (b) MVDR (c) RSBF



(a) (b) (c)

Fig.9 Actual AG under phase errors which has Gaussian distributions with variance from 0 degree to 10 degree for different *ka* (a) CBF (b) MVDR (c) RSBF

optimal points are selected for each *ka*, which can be observed from Fig.3.

In order to obtain a closed-form solution of the proposed robust beamforming technique, the expression of the optimization coefficient λ versus *ka* should be provided. Polynomial of different order is fitted and the fitting results are illustrated in Fig.4. It is observed that n=10 yields sufficient fitting accuracy. The optimization coefficient can thus be expressed as:

$$\lambda(ka) = p_1(ka)^n + p_2(ka)^{n-1} + \ldots + p_n(ka) + p_{n+1} \tag{20}$$

where p_i's, i=1,...,10 are the polynomial fitting parameters and are listed in Table I.

TABLE I. POLYNOMIAL FITTING OF OPTIMIZATION COEFFICIENT

Polynomial Order	Polynomial Fitting Parameters
10	[0.0049, -1.032, 0.9413, -4.8251, 15.2422, -30.5655, 38.5729, -29.1973, 11.8435, -1.9512, 0.1069]

The optimization coefficient equation (20), eigen-beams equation (6), noise covariance matrix equation (9), together with complex mode weighting equation (18) form a closed-form solution of the proposed RSBF.

To further evaluate the effectiveness of the proposed robust supergain beamformer, its performance is compared with CBF and MVDR of the fifth order, in the sense of AG and WNG, shown in Fig. 5. It is observed that compared with CBF, the proposed RSBF has larger AG (for *ka*=0.2 the AG for RSBF is 5.5dB larger than the CBF while as the *ka* becomes larger the difference maintains at about 2dB). In the aspect of WNG, the proposed RSBF has large WNG as CBF (about 2dB in *ka*=0.2 and above 9dB for other frequencies). Compared to MVDR, the WNG of the proposed RSBF is significantly larger, indicating a more robust performance over large array mismatch.

IV. SIMULATION

In this section, computer simulation is performed to validate the proposed RSBF and its advantages over both CBF and MVDR. Consider a circular array of N=12 omnidirectional hydrophones equally spaced on a circumference with a radius of 0.25m. The array is located on the equator of a rigid infinite cylinder. The 12 hydrophones allow sound pickup up to *ka* =6. Consider the sensor position and phase errors have Gaussian distributions with variance 0.03m and 3 degree, respectively. Mode weighting equation (18) and optimization coefficient equation (20) are used to form the proposed RSBF and both the theoretical and practical beam patterns are obtained for RSBF, CBF, and MVDR, shown in Fig.6 and Fig.7. Fig.6 plots the beampatterns for *ka*=0.2, representing the performance of the beamformers at low frequencies. MVDR has a large distortion from its theoretical pattern, which is also not realizable in practice. The proposed RSBF has identical theoretical pattern and pracitcal pattern, indicating its good robustness under large array mismatch. Fig.7 depicts the beampatterns for *ka*=1.0. For CBF, the large WNG makes the beamformer robust to array errors, and the obtained beam pattern has a -9dB sidelobe level and relatively wide mainlobe width. For MVDR the practical pattern again deviates from its theoretical value because of its poor robustness. For the proposed RSBF method, the good robustness makes the pattern identical to the theoretical one, while the obtained pattern has a lower sidelobe level (about -15dB) and relatively narrower mainlobe width than CBF.

The average AG over 100 Monte Carlo experiments for the above three beamformers under array position error and phase errors are examined, and the results are plotted in Fig. 8 and Fig. 9, respectively. In Fig. 8 (a) it is observed that CBF has good robustness for position errors from 0 cm to 6 cm. As *ka* increases, the AG increases, and the robustness of the CBF is degraded at high frequencies (*ka*>1.8). However, for low frequency range, the CBF performance is very poor (for *ka*=0.2 AG=0.8dB, for *ka*=0.6 AG=4dB, and for *ka*=1.0 AG=5.2dB). From Fig.8 (b) it is clear that the AG of the MVDR beamformer degrades fast as the position error becomes larger. For no error case, the MVDR beamformer can achieve 13.8dB array gain over the low frequencies. However, even for small position errors (i.e., 0.01m), the AG is smaller than 3dB, which is considered not acceptable. Fig. 8 (c) illustrates the performance of the proposed RSBF under array element position mismatch. It can be seen that the robustness of the proposed beamformer is very similar to CBF. In other words, the advantages of the CBF is well preserved. However, the proposed beamformer outperform the CBF in the AG (for *ka*=0.2 AG=6dB, for *ka*=0.6 AG=6dB, and for *ka*=1.0 AG=7.8dB). Fig.9 depicts the AG for three beamformers under phase errors, and similar conclusions can be drawn.

V. CONCLUSIONS

In this paper an optimal robust supergain beamforming technique is proposed which possess the advantages of CBF's good robustness and MVDR's good array gain. The derived analytical solution is in the form of diagonal loading approach yet the loading amount can be analytically expressed as a polynomial. Computer simulation verifies its robustness under large array mismatch.

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