Design of Robust Superdirective Beamformer for Circular Sensor Arrays

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***Abstract*—Two approaches to design robust superdirective beamformer for circular sensor arrays are presented in this paper. The first one is directly based on the eigen-decomposition of the circulant noise covariance matrix. Since the optimal solution can be accurately decomposed into sub-solutions denoted as eigenbeams and their robustness decreases with increasing of order number, a reduced-rank technique can be applied to obtain robust superdirectivity by truncating some error-intolerable high order eigenbeams. The results are all expressed in closed-form. The second approach is to formulate the beamformer as an optimization problem which can give an appropriate trade-off among directivity factor, sidelobe level and robustness. The robustness constraint parameter can be estimated from the performance of eigenbeams. Although it needs to be computed numerically, the computation efficiency can be improved using the properties of circular arrays. Experimental results show these two beamformers will provide good convenience to the practical implementation of robust high-order superdirectivity.**

Keywords—circular sensor arrays; high-order superdirectivity; robustness; optimal beamforming

I. INTRODUCTION

Superdirective beamforming method can provide high directivity, good angular discrimination and excellent wideband frequency-invariant performance using smaller aperture of array than its conventional counterpart, so it is attractive in many fields like sonar, radar, audio engineering and wireless telecommunication, etc [1-7]. But the theoretical good performance cannot be easily achieved in practice because of its unacceptable sensitivity to random errors from real sensor characteristics, the array configuration and the imprecise model etc. Many efforts have been done to minimize the effects of these errors and some approaches have been proposed, such as diagonal loading (DL) [8], white noise gain (WNG) [9] constraint and statistical approaches [10, 11], etc. Other model like the differential method [12, 13] and Taylor series framework (vector sensor technology) [14-17] provide different analytical solutions for superdirectivity, but they are not accurate enough making higher order superdirectivity still not available.

Circular array is widely used in many circumstances due to its simple configuration, no port-and-starboard-ambiguity and more importantly it can provide uniform beams along with 360° azimuthal directions. Consequently, its superdirectivity also have been received much attention. Besides the classical diagonal loading and white noise gain constraint methods, a new technique called modal beamforming becomes more and more attractive for circular arrays [3, 18-20]. However, this method is still not accurate for superdirectivity due to the errors caused by spatial sampling and series truncation.

Recently, Ma *et al.* provided an analytical and closed-form superdirective solution for circular arrays which is very inspired [21]. This approach is directly based on eigen-decomposition of the circulant noise covariance matrix and can be called eigenbeam decomposition and synthesis (EBDS) method. Although it provides good guidelines for practical implementation of high-order superdirectivity, the robustness in [21] needs to be discussed more deeply.

In this paper, the sensitivity function (SF) which is used to measure robustness is also decomposed into several closed form eigenbeams' sensitivity functions without any approximations utilizing the circulant properties of the noise covariance matrix, just like the optimal beampattern and its corresponding directivity factor. Based on the analyses of sensitivity function, two approaches are presented to design robust superdirective beamformer against errors for circular sensor arrays, particularly hydrophone arrays. The experimental results are shown to demonstrate the good performance of these two beamformers.

II. OPTIMAL BEAMFORMER WITH MAXIMUM DIRECTIVITY

The radius of the circular sensor array shown in Fig. 1 is a, and M omnidirectional sensors (e.g. hydrophones) are equally spaced. Unit-magnitude plane-wave impinges from direction (θ,ϕ), and the pressure received by the mth-sensor located at $(r_m,\phi_m,\pi/2)$ is

$$p_m(\theta,\phi)=e^{-ika\sin\theta\cos(\phi-\phi_m)} \tag{1}$$

where $i=\sqrt{-1}$, the wavenumber is $k=2\pi/\lambda$, λ denoting the wavelength. The manifold matrix will be

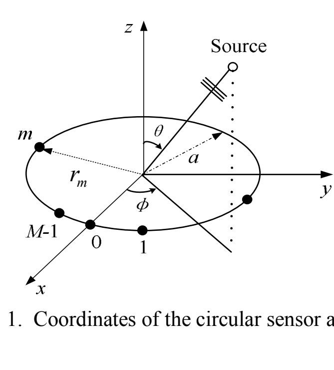

Fig. 1. Coordinates of the circular sensor array

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$$\boldsymbol{P}(\theta,\phi)=[p_0(\theta,\phi),p_1(\theta,\phi),\cdots,p_{M-1}(\theta,\phi)]^{\mathrm{T}} \quad (2)$$

Let $\boldsymbol{w}$ be the weighting vector, the beampattern will be

$$B(\theta,\phi)=\boldsymbol{w}^{\mathrm{H}}(\theta_0,\phi_0)\boldsymbol{P}(\theta,\phi) \quad (3)$$

where (θ_0,ϕ_0) is the preset steering direction.

Since the array gain is equivalent to the directivity factor (DF) of the array in an isotropic noise field, the classical weighting vector for maximum array gain is also optimum for maximum DF, which is [21, 22]

$$\boldsymbol{w}_{opt}=\alpha\boldsymbol{\rho}_n^{-1}\boldsymbol{P}(\theta_0,\phi_0) \quad (4)$$

where $\boldsymbol{\rho}_n$ is the normalized noise covariance matrix, $\boldsymbol{P}$ is the manifold vector, $\alpha=1/[\boldsymbol{P}^{\mathrm{H}}(\theta_0,\phi_0)\boldsymbol{\rho}_n^{-1}\boldsymbol{P}(\theta_0,\phi_0)]$ and the maximum directivity factor (DF) is coincidentally equal to $1/\alpha$. The beamformer with this kind of weighting vector is usually called minimum variance distortionless response (MVDR) beamformer.

The matrix $\boldsymbol{\rho}_n$ of the circular array in isotropic noise field is a circulant matrix which element is $\rho_s=\operatorname{sinc}(k\cdot\Delta r_s)$, Δr_s is the distance between the mth- and m'th-sensor, $s=|m-m'|$. The eigenvector and eigenvalue are [23]

$$\boldsymbol{v}_m=M^{-1/2}[1\quad e^{im\beta}\quad\cdots\quad e^{i(M-1)m\beta}]^{\mathrm{T}} \quad (5)$$

$$\lambda_m=\sum_{s=0}^{M-1}\rho_s e^{ism\beta} \quad (6)$$

respectively, and there will be $\boldsymbol{v}_m=\boldsymbol{v}_{M-m}^*$ and $\lambda_m=\lambda_{M-m}$. The eigenvevtors corresponding to different eigenvalues are mutually orthogonal. The optimal weighting vector in (4) can be changed to

$$\boldsymbol{w}_{opt}=\alpha\left[\sum_{m=0}^{M-1}\frac{1}{\lambda_m}\boldsymbol{v}_m\boldsymbol{v}_m^{\mathrm{H}}\right]\boldsymbol{P}(\theta_0,\phi_0) \quad (7)$$

Thus the maximum DF and the optimal beampattern can be both expressed in closed-form [21]

$$D=\sum_{m=0}^{M/2}D_m=\sum_{m=0}^{M/2}\frac{\varepsilon_m}{\lambda_m}|E_m(\theta_0,\phi_0)|^2 \quad (8)$$

$$B(\theta,\phi)=\sum_{m=0}^{M/2}B_m=\sum_{m=0}^{M/2}\alpha\frac{\varepsilon_m}{\lambda_m}\mathrm{Re}\left\{E_m^*(\theta_0,\phi_0)E_m(\theta,\phi)\right\} \quad (9)$$

The sensitivity function is always used to measure robustness of the beamformer, which is defined as [22]

$$T=\|\boldsymbol{w}\|^2 \quad (10)$$

Substitute (7) into (10), the SF can be stated as

$$T=\sum_{m=0}^{M/2}T_m=\sum_{m=0}^{M/2}\alpha^2\frac{\varepsilon_m}{\lambda_m^2}|E_m(\theta_0,\phi_0)|^2 \quad (11)$$

in which

$$E_m(\theta,\phi)=\boldsymbol{v}_m^{\mathrm{H}}\boldsymbol{P}(\theta,\phi)=\frac{1}{\sqrt{M}}\sum_{s=0}^{M-1}e^{-ism\beta}\cdot p_s \quad (12)$$

and $\varepsilon_m=1\,(m=0,M/2), \varepsilon_m=2\,(m=1,2,\cdots,M/2-1)$, M is even valued. It can be proved that $E_m(\theta,\phi)=(-1)^m E_{M-m}^*(\theta,\phi)$.

The B_m is called the mth-order eigenbeam with $DF=D_m$ and $SF=T_m$. Apparently, the final optimal beampattern and the maximum DF as well as the total SF are the sum of the eigenbeams, their DFs and SFs, respectively. Since the lower order eigenbeams are robust while higher order eigenbeams become more sensitive to errors, a reduced-rank technique can be adopted to obtain robust high-order superdirectivity which gives up a part of high-order eigenbeams. Although the final synthesized superdirective beampattern will have better directivity and narrower beamwidth, its sidelobe level (SL) is always a little bit high which should be reduced further.

III. ROBUST SUPERDIRECTIVE BEAMFORMER WITH SL CONSTRAINTS

Actually, the beampattern in (3) can be rewritten as

$$B(\theta,\phi)=\hat{\boldsymbol{\omega}}^{\mathrm{H}}(\theta_0,\phi_0)\hat{\boldsymbol{E}}(\theta,\phi) \quad (13)$$

where $\hat{\boldsymbol{E}}=[E_0,E_1,\cdots,E_{M-1}]^{\mathrm{T}}$, $\hat{\boldsymbol{\omega}}=[\omega_0,\omega_1,\cdots,\omega_{M-1}]^{\mathrm{T}}$ and the relationship between $\hat{\boldsymbol{\omega}}$ and $\boldsymbol{w}$ can be easily obtained

$$\boldsymbol{w}=\boldsymbol{V}\hat{\boldsymbol{\omega}} \quad (14)$$

in which $\boldsymbol{V}=[\boldsymbol{v}_0,\boldsymbol{v}_1,\cdots,\boldsymbol{v}_{M-1}]$. $\hat{\boldsymbol{E}}$ can be regarded as the new manifold vector.

Assume $\omega_{M-m}=(-1)^m\omega_m^*$, it is easy to know $\omega_m^*E_m(\theta,\phi)=[\omega_{M-m}^*E_{M-m}(\theta,\phi)]^*$. Thus, the final synthesized beampattern will be simplified to

$$\begin{aligned}B(\theta,\phi)&=\sum_{m=0}^{M-1}\omega_m^*E_m(\theta,\phi)\\&=\sum_{m=0}^{M/2}\varepsilon_m\,\mathrm{Re}\left\{\omega_m^*E_m(\theta,\phi)\right\}\\&=\mathrm{Re}\left\{\boldsymbol{\omega}^{\mathrm{H}}\left[\boldsymbol{\varepsilon}\circ\boldsymbol{E}(\theta,\phi)\right]\right\}\end{aligned} \quad (15)$$

where $\circ$ denotes the Hadamard product and $\boldsymbol{\omega}=[\omega_0,\omega_1,\cdots,\omega_{M/2}]^{\mathrm{T}}$, $\boldsymbol{\varepsilon}=[\varepsilon_0,\varepsilon_1,\cdots,\varepsilon_{M/2}]^{\mathrm{T}}$, $\boldsymbol{E}=[E_0,E_1,\cdots,E_{M/2}]^{\mathrm{T}}$. Define $B_m=\varepsilon_m\,\mathrm{Re}\left\{\omega_m^*E_m(\theta,\phi)\right\}$ as the mth-order generalized eigenbeam.

Applying the symmetry property of E_m, the expression of DF will be

$$\begin{aligned}D&=\frac{|\boldsymbol{w}^{\mathrm{H}}\boldsymbol{P}(\theta_0,\phi_0)|^2}{\boldsymbol{w}^{\mathrm{H}}\boldsymbol{\rho}_n\boldsymbol{w}}\\&=\frac{|\hat{\boldsymbol{\omega}}^{\mathrm{H}}\hat{\boldsymbol{E}}(\theta_0,\phi_0)|^2}{\hat{\boldsymbol{\omega}}^{\mathrm{H}}\hat{\boldsymbol{\Lambda}}_n\hat{\boldsymbol{\omega}}}\\&=\frac{\mathrm{Re}\left\{\boldsymbol{\omega}^{\mathrm{H}}\left[\boldsymbol{\varepsilon}\circ\boldsymbol{E}(\theta_0,\phi_0)\right]\right\}^2}{\boldsymbol{\omega}^{\mathrm{H}}\boldsymbol{\Lambda}_n\boldsymbol{\omega}}\end{aligned} \quad (16)$$

where $\boldsymbol{\Lambda}_n=\mathrm{diag}\left\{\varepsilon_0\lambda_0,\varepsilon_1\lambda_1,\cdots,\varepsilon_{M/2}\lambda_{M/2}\right\}$, and the SF will be

$$T=\|\boldsymbol{w}\|^2=\|\hat{\boldsymbol{\omega}}\|^2=\boldsymbol{\omega}^{\mathrm{H}}(\boldsymbol{\varepsilon}\circ\boldsymbol{\omega}) \quad (17)$$

The design of robust superdirective beamformer with SL and robustness constraints can be formulated as an optimization problem

$$
\begin{aligned}
&\min_{\boldsymbol{\omega}} \quad \boldsymbol{\omega}^{\mathrm{H}} \boldsymbol{\Lambda}_n \boldsymbol{\omega}, \\
&\text{s.t.} \quad \mathrm{Re}\left\{\left[\boldsymbol{\varepsilon} \circ \boldsymbol{E}(\theta_0, \phi_0)\right]^{\mathrm{H}} \boldsymbol{\omega}\right\} = 1, \\
&\qquad \boldsymbol{\omega}^{\mathrm{H}} (\boldsymbol{\varepsilon} \circ \boldsymbol{\omega}) \le \sigma, \\
&\qquad \mathrm{Re}\left\{\left[\boldsymbol{\varepsilon} \circ \boldsymbol{E}(\theta_0, \phi_j)\right]^{\mathrm{H}} \boldsymbol{\omega}\right\} \le \delta_j, \\
&\qquad (\phi_j \in \phi_{SL},\ j = 1, \cdots, N_{SL})
\end{aligned}
\tag{18}
$$

in which σ is the specified upper bound of the SF, and δ_j is the prescribed sidelobe peak value. ϕ_j is the discrete N_{SL} directions in the sidelobe areas. Only the horizontal beampattern ($\theta_0 = 90^\circ$) is considered here. The beamformer obtained by (18) is robust due to the constraint of sensitivity function, and the sidelobe levels will be controlled properly. Actually, the eigenbeam can be regarded as an indicator of robustness. From the performance of different eigenbeams, the errors of the actual array can be estimated and the upper bound of the SF in (18) can be determined. Note α will be changed with the number of the retained eigenbeams to normalize the beam response in the steering direction, which will be

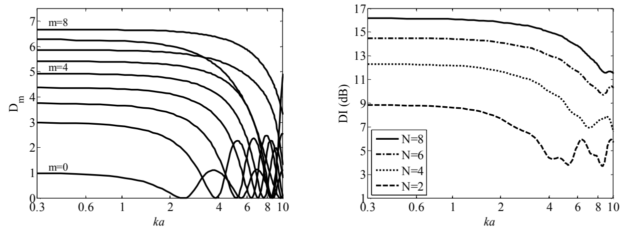
$$
\alpha = 1 / \left[\sum_{m=0}^{N} \frac{\varepsilon_m}{\lambda_m} \left|E_m(\theta_0, \phi_0)\right|^2\right] \tag{19}
$$

Unlike the closed-form weighting vector in (9) with element $\omega_m = \alpha E_m(\theta_0, \phi_0) / \lambda_m$, the weighting vector in (18) can be only computed using numerical methods, e.g. second-order cone programming (SOCP) [24]. This kind of weighting vector should be expanded to $\hat{\boldsymbol{\omega}}$ using $\omega_{M-m} = (-1)^m \omega_m^*$, and then it can be transformed into the classical form applying (14). Since the dimension of the weighting vector in (18) is almost the half of the sensor number, the computation load can be reduced.

IV. SIMULATION

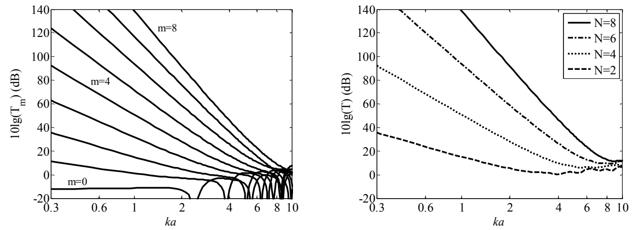
The DFs of different eigenbeams and the total DI ($DI = 10\lg D$ (dB)) calculated by (8) for different maximum order N of the retained eigenbeams are shown in Fig. 2. The preset steering direction is $(\theta_0, \phi_0) = (\pi/2, \pi)$. The numbers adjacent to the lines in Fig. 2(a) denote the corresponding eigenbeam orders. In the high frequency band, each DF line is undulate and shows some zeros at several frequencies. By contrast, in low frequency range which we are interested in, the DFs of the circular array increase stably along with the order's increase but has a decreasing trend with frequency. Since the total DI is the accumulation of the DFs of the selected eigenbeams with orders from 0 to N, their properties of changing with order number and frequency are similar. It is noteworthy that truncating more error-intolerable higher order eigenbeams will make the DF decrease, but in the following section it is found the robustness is improved. As the frequency tends to 0, the DFs will approach the limit values.

Fig. 3 displays the sensitivity function evaluated by (11) versus ka. Without loss of generality, the scalar α is assumed to be 1. In high frequency range, the SFs of different eigenbeams also show some dips just like DFs in Fig. 2. As the



(a) DFs of different eigenbeams (b) Total DI

Fig. 2. Directivity factors of eigenbeams and total directivity index of the circular array



(a) SFs of different eigenbeams (b) Total SF

Fig. 3. Sensitivity functions of eigenbeams and total sensitivity function of the circular array

frequency decreases to 0, the robustness will be worse. It has been pointed out that truncating the high order eigenbeams which is more sensitive to errors leads to the degraded DF, but the simultaneously decreased SF means the improved robustness. Thus the useful reduced-rank way can be adopted to compromise between directivity and robustness. It is observed that the SF of high order is much larger than the low one. As the summation of them, the total SF is nearly equal to the SF of the maximum order N in these retaining eigenbeams, which is

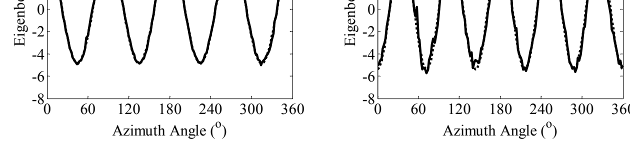
$$
T = \sum_{m=0}^{N} T_m \approx T_N \tag{20}
$$

This approximation is available in the low frequency band and not concerned with the dips. No matter how many eigenbeams are selected, the robustness of the final synthesized beampattern is determined by the highest order eigenbeam. When it is damaged, the final beam will be affected seriously. In practice, special care has to be taken when choosing N.

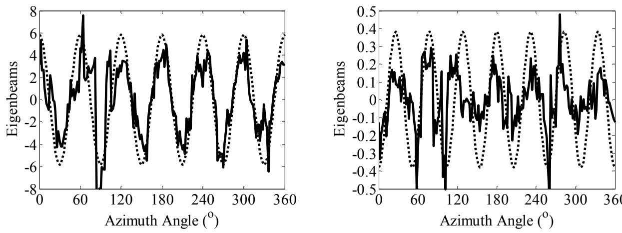
V. EXPERIMENTAL RESULTS

An experimental 16-hydrophone acoustic array without baffle was made and the radius is 0.25m. Only the horizontal eigenbeams were extracted from experimental data acquired in an anechoic water tank.

The practical and theoretical eigenbeams ranging from 4th- to 7th-order at ka=1.16 are shown in Fig. 4. It is seen that the experimental eigenbeams with order smaller than 6 exhibit

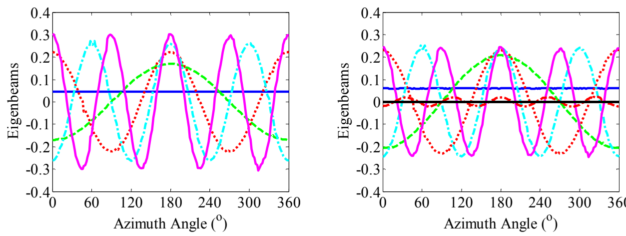


(a) m=4 (b) m=5



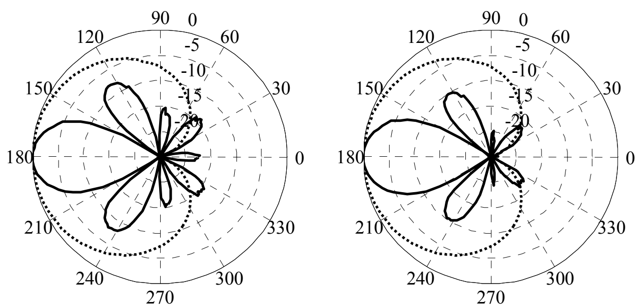
(c) m=6 (d) m=7

Fig. 4. Practical (solid line) and theoretical (dotted line) eigenbeams at ka=1.16



(a) Without constraints (b) With SL and robustness constraints

Fig. 5. Practical eigenbeams at ka=1.16



(a) Without constraints (b) With SL and robustness constraints

Fig. 6. Superdirective beampatterns (solid line) and conventional beampattern (dotted line) at ka=1.16

perfect agreement with the corresponding theoretical ones. By comparison, the higher order eigenbeams are all corrupted more or less by errors and the distortions increase with the order number which is consistent with the analyses of SF. The 0th- to 3rd- and 8th-order eigenbeams are not displayed here for compactness. This interesting feature indicates the existed errors cannot be tolerated by the eigenbeams with order larger than 5, which is to say the maximum SF should be about $10\lg(T_5)=65.2\text{dB}$ from Fig. 3(a). After using the scalar α to normalize the beam response in the steering direction, the maximum SF will be changed to 40.9dB and the 0th- to 4th-order normalized eigenbeams are shown in Fig. 5(a). For conservative estimate, the upper bound of the SF is set to be equal to the normalized 4th-order eigenbeam which is $\sigma = T_4 = 10^{21.5/10}$.

Fig. 5(b) depicts the generalized eigenbeams with SL and robustness constraints, which show large amplitude difference from the normalized eigenbeams in Fig. 5(a), especially the 6th- to 8th-order eigenbeams are so small that can be neglected. The final superdirective beampattern is synthesized using the properly retained eigenbeams, which are compared with conventional beampattern in Fig. 6. As listed in Table I, the superdirective beampatterns have the higher directivity index

TABLE I. THE DI, HPBW AND SL OF PRACTICAL SUPERDIRECTIVE AND CONVENTIONAL BEAMPATTERNS

***ka*=1.16**	**Superdirective (unconstrained)**	**Superdirective (constrained)**	**Conventional**
DI(dB)	12.2	12.2	4.2
HPBW(°)	31.2	32.6	115.9
SL(dB)	-7.8	-10.0	--

(DI), narrower half-power beamwidth (HPBW) and lower SL. The SL of unconstrained superdirective beampattern is still unacceptably high which should be reduced. For comparison, the SL of the constrained superdirective beampattern is controlled under the desired level -10dB and the DI is not changed, although the HPBW is widened a little.

VI. CONCLUSION

Two robust superdirective beamformers for circular sensor arrays are presented. The first one is directly based on the eigen-decomposition of the circulant noise covariance matrix and the solutions are all expressed in closed-form without any approximations. The robust superdirective beampattern is obtained using a reduced-rank technique, but its SL is still high. The second one is a simple optimization problem which can also control the SL. In practice, the appropriate robustness parameter can be estimated according to the performance of the different eigenbeams. Therefore, this beamformer is more flexible. Although the solution is not in closed-form, the computation complexity is simplified. Experimental results demonstrate these two robust beamformer can provide good convenience to the practical implementation of robust high-order superdirectivity.

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