Nonlinear spectra of shallow water waves

J.-P. Giovanangeli, C. Kharif, N. Raj and Y. Stepanyants

Abstract—The problem of spectra interpretation of nonlinear shallow water waves is studied in terms of interacting Korteweg–de Vries (KdV) solitons and quasi-linear wavetrains. The method of data processing of random wave field is suggested and illustrated by an example. The soliton component obscured in the random wave field can be determined either on the basis of the inverse scattering method or by direct numerical solution of the KdV equation. The distribution function of number of solitons on amplitudes was constructed for the illustrative example. The relevance of the suggested approach to field measurements is discussed.

Index Terms—shallow water, wind waves, random wave field, wave spectra, solitons, numerical modelling, laboratory experiments.

1 Introduction

Traditional approach in the problem of wind wave study is based on the analysis of Fourier spectra and their peculiarities. An interesting and useful information on wind waves has been obtained by such technique. However, Fourier analysis of waves provides researchers with only some pieces of objective information, whereas many important features of random wave fields remain obscured. One of the serious obstacles making the Fourier analysis ineffective in application to surface oceanic waves is the nonlinear character of such waves, whereas the Fourier analysis is a linear operation most relevant to systems obeying the superposition principle.

Osborne with co-authors (see, e. g., [5], [6], [7], [8] and references therein) developed the method of nonlinear spectral analysis of shallow water waves described by the Korteweg–de Vries (KdV) equation. Osborne's approach is based on the application of the inverse scattering method (ISM) to the analysis of random field data in the one-dimensional space domain with periodic boundary conditions. The main idea of his approach is to interpret the complex initial disturbance in terms of a set of elliptic functions. These functions can be considered as the nonlinear eigenmodes, which are conserved in the process of wave field evolution in contrast to the linear sinusoidal eigenmodes, which are permanently change due to nonlinear interaction between them. Hence, the nonlinear wave spectrum calculated on the basis of elliptic functions is invariant in time, whereas the Fourier spectrum of trigonometric functions varies. In the case of small-amplitude perturbations the spectrum of nonlinear eigenmodes naturally reduces to the Fourier spectrum of linear modes. However, the mathematical and numerical tools used for the calculation of nonlinear eigenmodes is not so simple as in the linear case and, apparently, they are impractical, especially for applied mariner engineers.

Here we propose a bit different approach to the analysis of random water waves that is also based on the application of ISM and is similar to Osborne's approach. The essential feature of our approach is that we interpret the initial wave field as an ensemble of interacting solitons and quasi-linear ripples rather than the set of elliptic eigenmodes. The idea is illustrated by the example of shallow water waves described by the classical KdV equation with random initial data. If one takes some portion of the random field data (which

- *J.-P. Giovanangeli and C. Kharif are with the Laboratoire IRPHE–IOA, Campus Luminy, Case 903 13288, Marseille, Cedex 9, France.*
  E-mail: Kharif@irphe.univ-mrs.fr
- *N. Raj and Y. Stepanyants are with the University of Southern Queensland, Toowoomba, Australia.*
  E-mail: Yury.Stepanyants@usq.edu.au

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should be long enough), the number of solitons, their amplitudes, speeds, characteristic durations, etc., can be calculated by means of ISM (e.g., by solving numerically the associated Sturm–Louiville problem) or by direct numerical simulation of the corresponding KdV equation. In both cases, the numerical codes are currently well developed and easily accessible. In the meantime, knowledge of number of solitons obscured in the random wave field, their parameters and statistics is a matter of independent interest. Different geographic zones where surface or internal solitons appear regularly can be characterized by the specific soliton statistics, which may depend in turn on a season. Below we describe our approach and illustrate it by the example of processing of quasi-random wave filed.

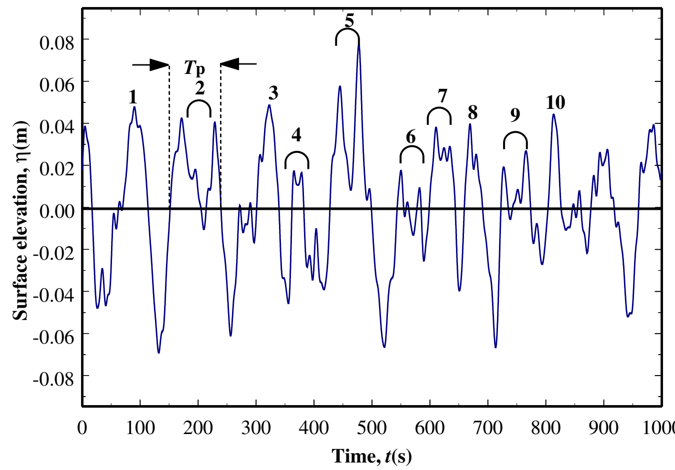

Fig. 1. The model example of artificially generated random time series of surface waves in shallow-water basin. The labelled pulses are explained in the text.

2 KdV MODEL AND DATA PROCESSING

Consider the time series of surface wave record in a fixed point x_0 of a shallow-water basin. The elevation of the water level at this point can be treated as the known function of time: $\eta(x_0, t) = f(t)$ where $f(t)$ is some regular or random function. A typical example of a random record of surface waves is shown in Fig. 1 (for the illustrative purposes these data were artificially generated by means of a generator of random numbers).

For the description of further evolution in space of the data recorded at point x_0, we use the so-called timelike KdV equation (TKdV Eq.) [7] that is actually a version of the usual KdV equation in which the time and space variables are inter exchanged:

$$\eta_x + \eta_t/c_0 - \alpha\eta\eta_t - \beta\eta_{ttt} = 0, \tag{1}$$

Here $c_0 = \sqrt{gh}$; $\alpha = 3/(2c_0 h)$; $\beta = h^2/(6c_0^3)$ with g being the acceleration due to gravity and h being an unperturbed water depth.

In the process of evolution of an initial perturbation one can expect an emerging of solitons with different amplitudes and phases (i.e., time markers of soliton maxima in the wave record at the given point of observation). Soliton solution to Eq. (1) has the form (see, e.g., [1], [3], [9]):

$$\eta(x, t) = A\,\mathrm{sech}^2\,\frac{t - x/V}{T}, \tag{2}$$

where the soliton duration T and velocity V are related to the amplitude A:

$$T = \sqrt{\frac{12\beta}{\alpha A}}; \quad V = \frac{c_0}{1 - c_0\alpha A/3} \approx c_0\left(1 + \frac{c_0\alpha A}{3}\right).$$

The approximate formula holds for $c_0\alpha A \ll 1$.

As it was mentioned in [7], "it may not be possible to observe solitons in real space" because of numerous nonlinear interactions of solitons both with each other and with chaotic radiation background component. However, "it would be naive to conclude that solitons are not present or that their dynamics are not important" in the evolution of the initial perturbation. Inasmuch as solitons are very stable with respect to interaction with others wave perturbations and influence of external effects (such as viscosity, inhomogeneity, etc), it is a matter of interest to extract them from the irregular components of a wave field and to describe their statistical properties and contribution to the total wave energy.

The solution of this problem can be done by the following way. Let us consider a very long portion of the measurement data of surface perturbation at any given point x_0. The characteristic duration of this portion T_p is assumed to be much greater than the typical soliton time scale T. Let us represent a perturbation with the help of some dimensionless random function $\varphi(t)$:

$$\eta(0,t) = U\varphi(t/T_p), \tag{3}$$

where U is the characteristic wave "amplitude", e.g., the maximum value of the perturbation $\eta(0,t)$ within the considered portion of the data.

By means of the transformation

$$u = \frac{\eta}{U}, \quad \xi = -\frac{\alpha U}{T_p}x, \quad \tau = \frac{1}{T_p}\left(t - \frac{x}{c_0}\right),$$

Eq. (1) and corresponding "initial" perturbation (3) (which is, in fact, the perturbation at the boundary $x = x_0$) can be reduced to the standard form [3]:

$$u_\xi + uu_\tau + \frac{1}{\sigma^2}u_{\tau\tau\tau} = 0$$
$$u(0,\tau) = \varphi(\tau)$$

with one dimensionless parameter σ known in the oceanography as the Ursel parameter and defined as $\sigma^2 = \alpha U T_p^2/\beta$.

As has been mentioned above, we consider the case when the duration of the initial perturbation, T_p, is long enough, so that $\sigma^2 \gg 1$. In this case the number of solitons obscured in the initial perturbation is also very large, in general. In this case, it is reasonable to apply statistical methods for their description with the help of a distribution function $f(A)$. This function determines the number of solitons dN within the amplitude interval (A, dA) [3]:

$$dN = f(A)dA. \tag{4}$$

According to [3], for large σ the distribution function can be calculated by means of the formula:

$$f(A) = \frac{\sigma}{4\pi\sqrt{3U}}\int\limits_L \frac{d\tau}{\sqrt{2U\varphi(\tau) - A}}, \tag{5}$$

where the integration domain L is determined by the condition $2U\varphi(\tau) > A$.

The soliton amplitudes are distributed in the interval $0 < A < 2U\max[\varphi(\tau)]$, and their total number is calculated by the formula

$$N = \int\limits_0^\infty f(A)\,dA = \left(\frac{\sigma}{\pi\sqrt{6}}\right)\int\limits_{\varphi(\tau)\,>\,0}\sqrt{\varphi(\tau)}\,d\tau. \tag{6}$$

Hence, when $\sigma \gg 1$, the total number of solitons is determined only by those intervals in the τ-axis, where the function $\varphi(\tau)$ is positive!

As it is well known, the KdV equation possesses the infinite number of conserved quantities I_n [3]. One of them,

$$I_2 = \int\limits_{-\infty}^{+\infty}\eta^2(x,t)\,dt,$$

is proportional to the wave energy and is therefore a matter of special physical interest (we call this quantity just the "energy" for the sake of simplicity).

A fraction of energy in the nonsoliton component of the initial perturbation to the total energy of the perturbation can be determined by means of the formulae (18.27), (18.28) and (19.8) of Ref. [3], *viz.*:

$$\frac{I_2^{ns}}{I_2^{tot}} = \frac{\int\limits_{\eta(0,\tau)\,<\,0}\eta^2(0,\tau)\,d\tau}{\int\limits_0^{T_p}\eta^2(0,\tau)\,d\tau}. \tag{7}$$

On the other hand, the total energy of the soliton component in the wave field can be readily calculated if soliton amplitudes are known:

$$I_2^{sol} = \frac{4}{3}\sqrt{\frac{3\beta}{\alpha}}\sum_{k=1}^{N}A_k^{3/2} = \frac{4h}{3\sqrt{3g}}\sum_{k=1}^{N}A_k^{3/2}. \tag{8}$$

To illustrate the idea of this approach, we consider below the example of data processing for the artificially generated random time series shown in Fig. 1.

2.1 Data processing for the model case

2.1.1 The model spectrum and range of its validity

Let us apply now the approach described above to the random initial perturbation artificially generated and presented in Fig. 1. This perturbation has been obtained by means of the inverse Fourier transform of the series of 128 Fourier harmonics having random phases in the interval [0, 2π] and amplitudes distributed in accordance with the formula (see Fig. 2)

$$S(\omega) = S_0(\omega + \omega_0)^{-2}\tanh\frac{\omega}{\delta\omega},$$

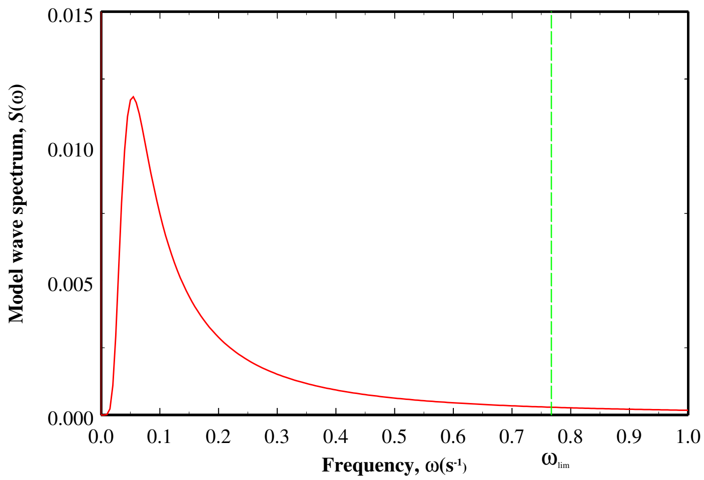

Fig. 2. The model amplitude spectrum of the random wave field shown in Fig. 1.

where $S_0 = 2 \cdot 10^{-4}$ m, $\omega_0 = 2\pi \cdot 10^{-2}$ s^{-1}, $\delta\omega = 2.5 \cdot 10^{-2}$ s^{-1}.

For the creation of the quasi-random wave field it was taken only the low-frequency and most energetic part of this spectrum, $0 \le \omega \le \omega_{lim}$, where $\omega_{lim} = 0.767$ s^{-1} (see the vertical dashed line in Fig. 2). The limiting frequency, ω_{lim}, was chosen for the following reasons. The dispersion relation for the TKdV Eq. (1) is

$$k = \frac{\omega}{c_0} + \frac{h^2}{6c_0^3}\omega^3. \tag{9}$$

As is well known, this dispersion relation usually represents the approximation of more complex dispersion relations of real physical systems, e.g., for water or plasma waves, etc. The range of validity of Eq. (9) is restricted by the requirement that the second term in the right-hand side of Eq. (9) is relatively small in comparison with the first one (see, e.g., [1], [3], [9]); this gives:

$$\omega \ll \omega_{cr} \equiv \sqrt{\frac{6c_0^2}{h^2}} = \sqrt{6\frac{g}{h}}. \tag{10}$$

In our case $\omega_{cr} \approx 7.67$ s^{-1}. The condition (10) can be satisfied if we put $\omega_{lim} = \omega_{cr}/10 = 0.767$ s^{-1}.

2.1.2 Data processing of the model initial perturbation

Let us consider now quasi-random wave field shown in Fig. 1 and apply to its processing the approach described above. Each positive hump enumerated in the figure was studied separately by means of the developed numerical code for the TKdV Eq. (1). The numerical code was based on the explicit finite-difference scheme of the second-order accuracy both on the spatial and temporal variables [2]. The theoretical analysis of the numerical scheme shows that it is conditionally stable provided that $\Delta x \sim \Delta t^3$, where Δx and Δt are the spatial and temporal mesh steps. The code works very effectively and fast so that the results of pulse fissions onto solitons were obtained very quickly on a PC computer. The typical picture of disintegration of one of the pulses (the pulse No 5 in Fig. 1) is shown in Fig. 3.

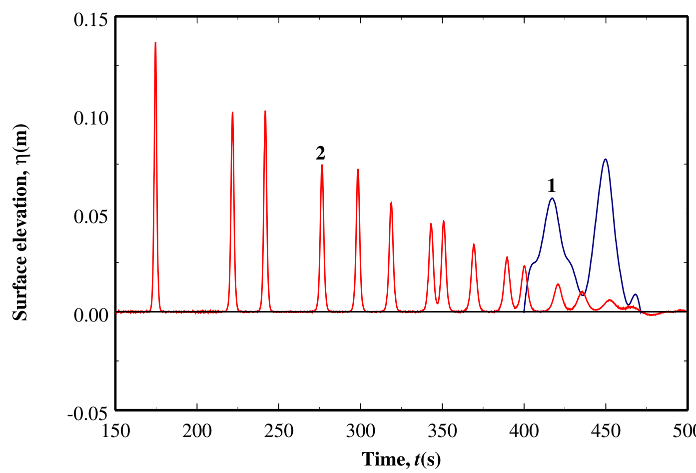

Fig. 3. Disintegration of one of the initial random pulses (line 1) on solitons (line 2).

The number of emerging solitons and their parameters can be easily calculated. To avoid errors in the determination of soliton parameters, the calculations were conducted until each soliton was well separated from others, so that their fields were not overlapped in the vicinity of their maxima. This procedure of pulse processing and determination of soliton parameters was undertaken for each pulse shown in Fig. 1. As the result, it was obtained a large number of solitons of different amplitudes; their total number for all pulses was $N_s = 73$, which is enough for the illustrative purposes. The solitons were collected into 15 groups according to their amplitudes. This allows us to construct a histogram of numbers of solitons against their amplitudes. The histogram can be considered as the model of the distribution function $f(A)$ of the density of number of solitons on amplitudes (see Eq. (5)). The histogram obtained and the model distribution function for the considered illustrative example are shown in Fig. 4. Figure 5 shows the cumulative distribution function, i.e. the total number of solitons whose amplitudes are not greater than the given value.

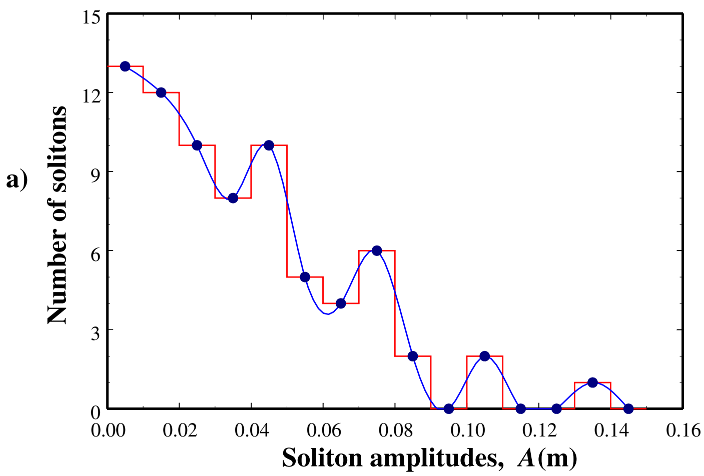

Fig. 4. The histogram of soliton distribution (piece-linear line) and model distribution function (smooth line) for the illustrative example.

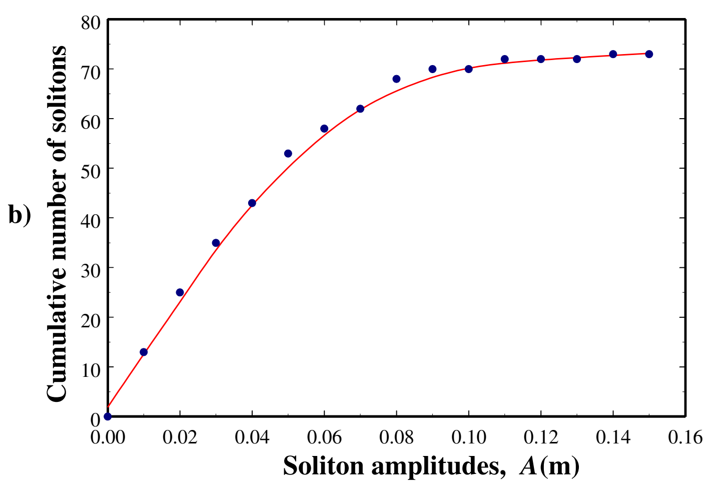

Fig. 5. The cumulative distribution function versus soliton amplitudes.

In the similar way one can obtain the distribution function and cumulative distribution function of solitons on their amplitudes for real wave fields. One of the rear attempts to obtain the distribution function of solitons from the real experimental data has been undertaken in paper [4].

3 Discussion and Conclusion

To analyze the long random time series of water waves in shallow basins we propose the approach which differs from the traditionally used Fourier analysis. Our approach is based on the extraction of obscured solitons from the complex quasi-random wave fields and construction of histograms of solitons at different points of observation. The histograms can be considered as the experimental analog of smooth distribution functions of number of solitons on their amplitudes (see Figs. 4 and 5). For well-developed nonlinear perturbations the soliton component in a wave field is dominant according to theoretical conceptions (see, e.g., [3]). As is well known, the number and individual parameters of solitons preserve in the conservative systems, hence, the distribution function (or the histograms of solitons) should be the same at different points of observation along the travelled path, whereas the Fourier spectrum changes significantly due to nonlinear character of surface waves causing nonlinear energy exchange between Fourier harmonics. A small dissipation may cause a gradual decay of solitons and slow change of histograms. Therefore one may expect the gradual histogram distortion in space, whereas the main features of histograms may conserve on long distances.

Our approach can provide some additional interesting information about the energy distribution in natural wave fields, such as the relationship between the soliton and nonsoliton components of the perturbation. It may also indicate on the existence of external sources or sinks of wave energy in the basin, as well as shed light on their intensity.

To determine the soliton number and amplitudes from the random time series, we used here the direct numerical calculations of initial data within the framework of the TKdV equation. The existing numerical codes (see, for example, [2]) allow one to obtain results of computations fairly quickly in the convenient form. However, it is not only the method which can be used; the numerical solution of the eigenvalue problem for the TKdV equation within the framework of ISM can be also convenient and useful (for details of ISM and related eigenvalue problem see, for example, [1], [3], [9]).

Our experience with the application of the described approach shows that there is no problem with the determination of the number and amplitudes of well-developed solitons whose amplitudes are sufficiently big. However, it is not so simple to determine the parameters of solitons whose amplitudes are very small (see, e.g., Fig. 3). In other words, the uncertainty in determination of amplitudes of very small solitons is rather high. Meanwhile, the model example considered above shows that the total number of such small-amplitude solitons may be relatively big (see Fig. 4).

In conclusion, we would like to emphasize that the approach developed here is applicable to the wave systems which can be described by the KdV-type equation, e.g., to surface and internal shallow-water waves. Its generalization to the deep-water waves which are described by the nonlinear Shrödinger equation is an interesting and challenging problem.

The full paper with more detailed explanation of the suggested approach and additional examples will be published elsewhere.

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