TEL AVIV UNIVERSITY אוניברסיטת תל-אביב
FACULTY OF ENGINEERING הפאקולטה להנדסה

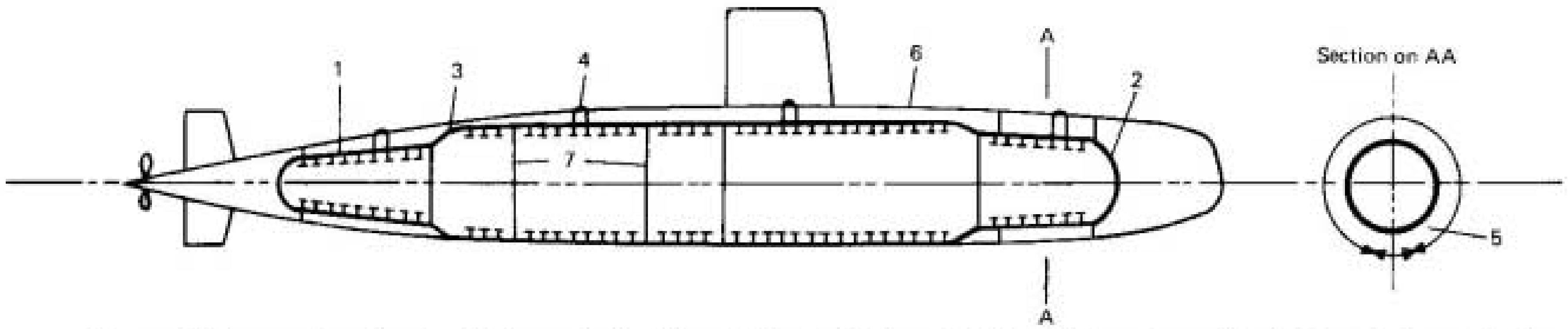
Lesson 10 – Bending and stability of cylindrical shells with application to submarine design

Outline

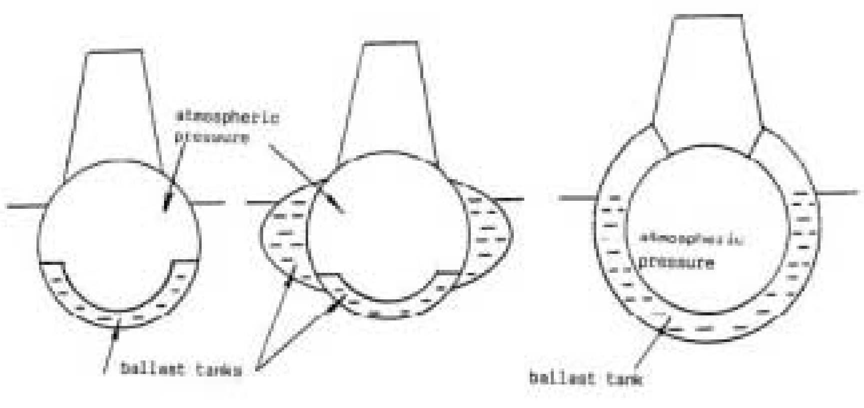
1. **Elements of shell theory. Differential equations for cylindrical shells.**
2. **Main stresses conditions of shells**
   a) **Membrane stresses**
   b) **Bending stresses**
   c) **General case**
3. **Bending of a reinforced cylindrical shell. Reduction to the beam on an elastic foundation**
4. **Interframe shell instability. Mises formula.**
5. **Stability of reinforced cylindrical shell. Reduction to the orthotropic shell model.**
6. **Influence of inelasticity and shape imperfections on the critical pressure. Design charts.**

שנת הלימודים תשס״ב
סמסטר אביב

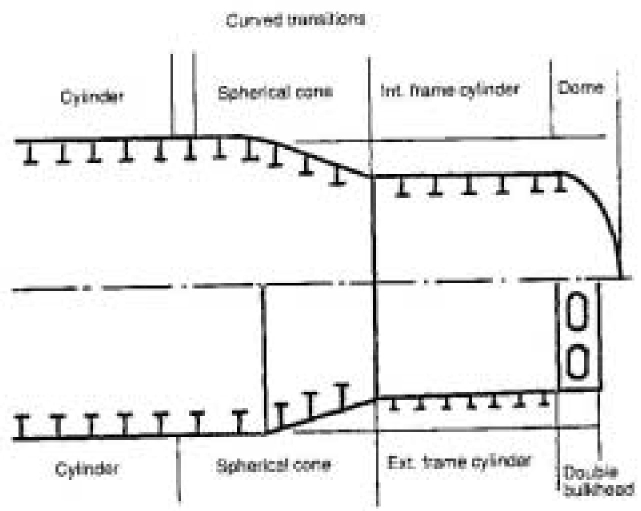
Submarine hull construction



Structural features of a submarine. 1. Pressure-hull plating and framing; 2. Domed bulkhead; 3. Curved transition; 4. Hatch; 5. External ballast tank; 6. Hull plating; 7. Main transverse bulkheads.

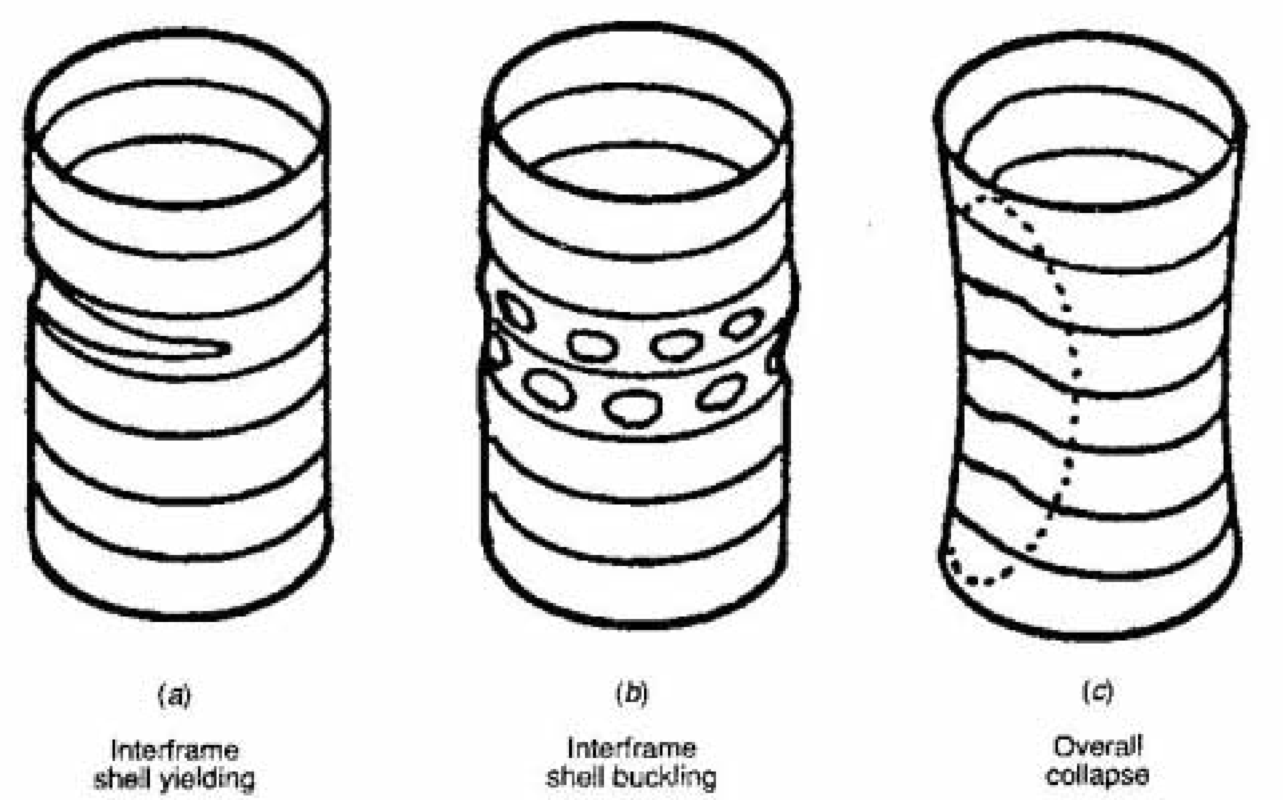


(a) Single hull (b) intermediate hull (c) double hull



Change of diameter and closure

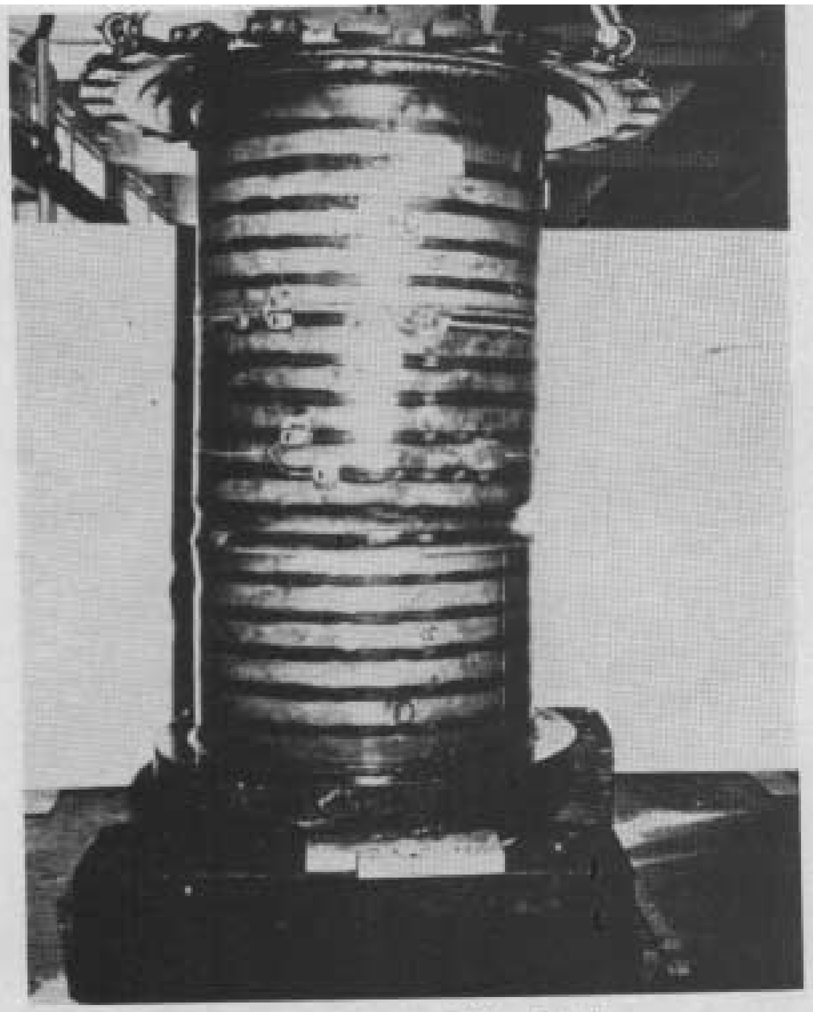
Pressure hull faillure modes



Failure modes of ring stiffened thin circular cylinders

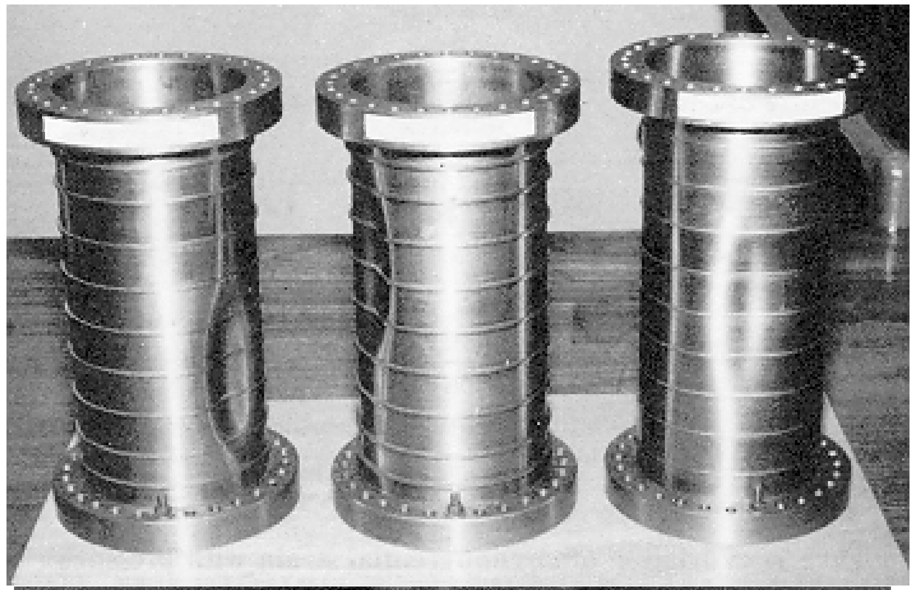
Examples of shell yield and shell buckling (models)

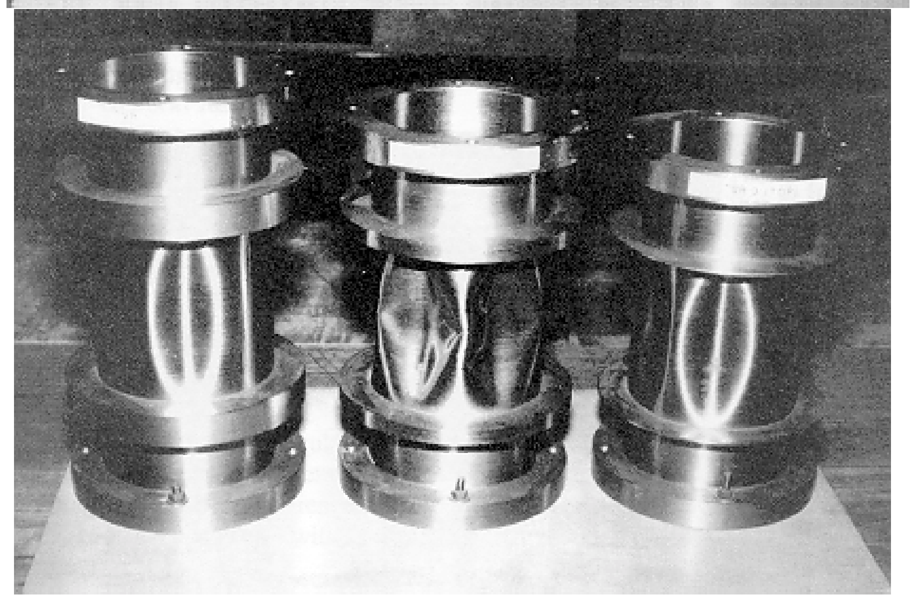
Failure by buckling of shell

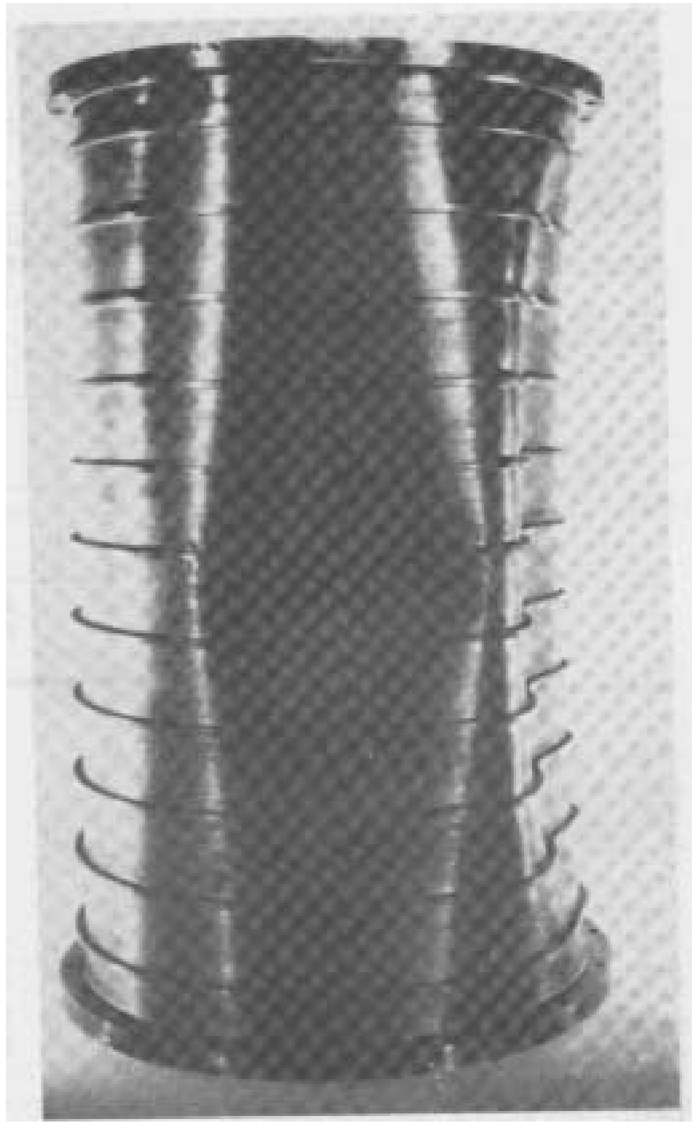


Failure by yielding of shell

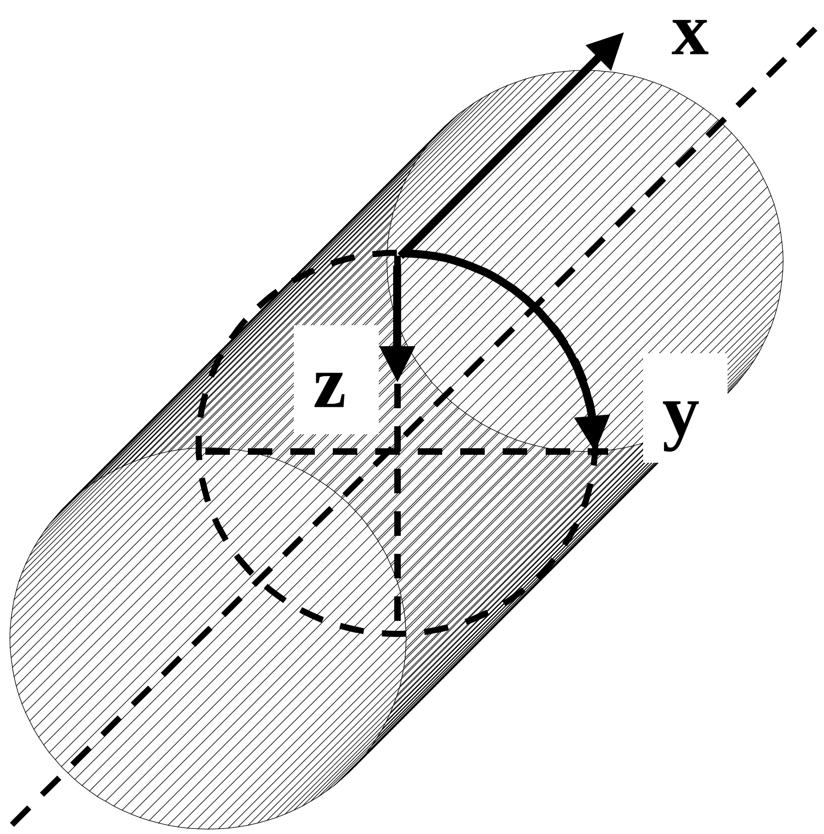
Examples of overal shell buckling (models)



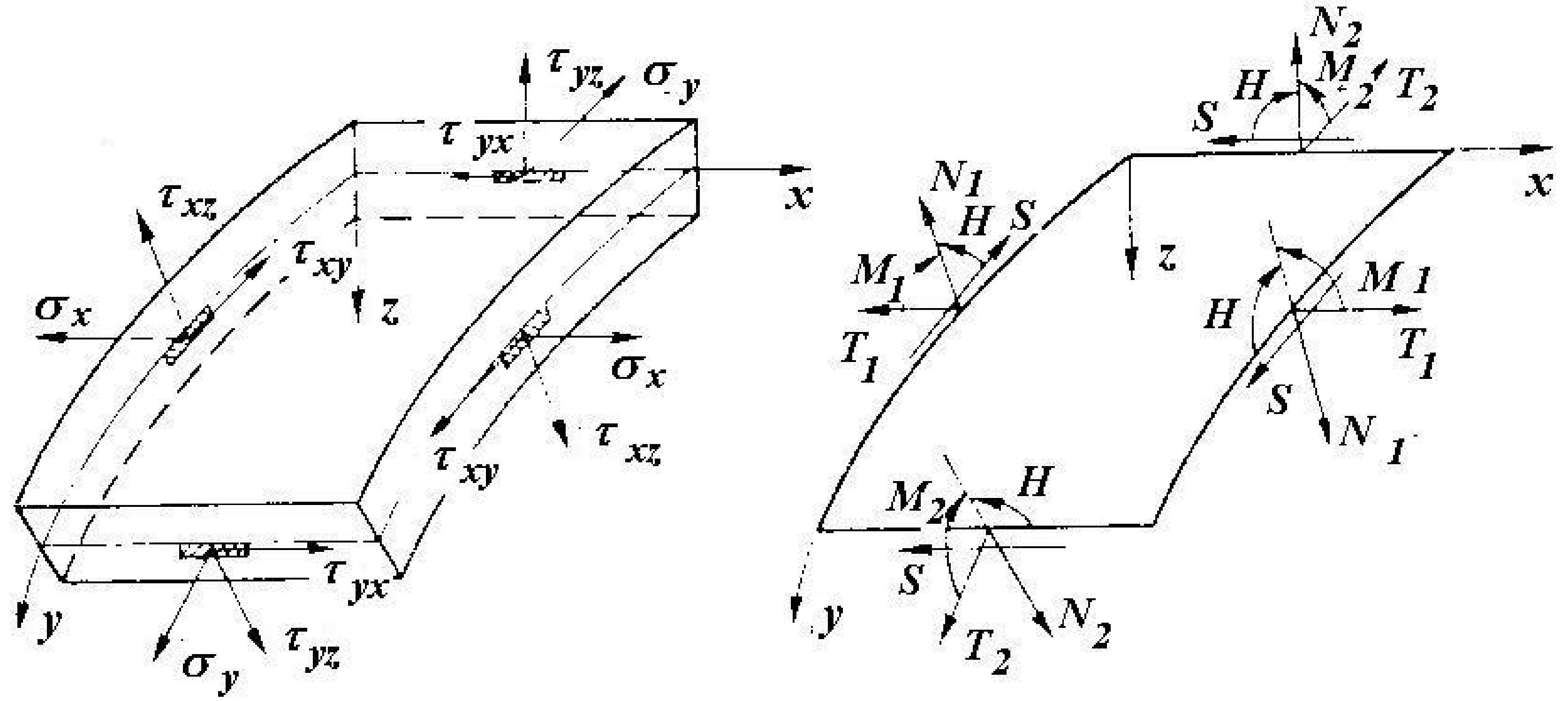




Coordinate system for cylindrical shell



Forces (per unit length) acting on the shell element



Eight unknown forces: T_1 T_2 M_1 M_2 N_1 N_2 S H

Equilibrium equations of shell in terms of forces

$$\frac{\partial T_1}{\partial x} + \frac{\partial S}{\partial y} = 0; \quad \frac{\partial S}{\partial x} + \frac{\partial T_2}{\partial y} - \underline{\frac{N_2}{R}} = 0$$

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_2}{\partial y} + \underline{\frac{N_2}{R}\frac{\partial w}{\partial y}} + T_1\frac{\partial^2 w}{\partial x^2} + 2S\frac{\partial^2 w}{\partial x\,\partial y} + T_2\left(\frac{\partial^2 w}{\partial y^2} + \frac{1}{R} + \frac{w}{R^2}\right) + p = 0;$$

$$N_1 = \frac{\partial M_1}{\partial x} + \frac{\partial H}{\partial y}; \quad N_2 = \frac{\partial M_2}{\partial y} + \frac{\partial H}{\partial x}.$$

Eight unknown forces: *T_1 T_2 M_1 M_2 N_1 N_2 S H*

Relation between internal forces and displacements/midplane deformations-orthotropic shell

$$T_1 = \frac{E_x t}{1-\mu_{xy}\mu_{yx}}\left(\varepsilon_x^0 + \mu_{xy}\varepsilon_y^0\right);$$

$$T_2 = \frac{E_y t}{1-\mu_{xy}\mu_{yx}}\left(\varepsilon_y^0 + \mu_{yx}\varepsilon_x^0\right); \quad S = G_{xy} h \gamma_{xy}^0;$$

$$M_1 = -D_1\left[\frac{\partial^2 w}{\partial x^2} + \mu_{xy}\left(\frac{\partial^2 w}{\partial y^2} + \frac{w}{R^2}\right)\right];$$

$$M_2 = -D_2\left[\left(\frac{\partial^2 w}{\partial y^2} + \frac{w}{R^2}\right) + \mu_{yx}\frac{\partial^2 w}{\partial x^2}\right]; \quad H = -2D_t\frac{\partial^2 w}{\partial x\,\partial y},$$

$\varepsilon_x^0 \ \varepsilon_y^0 \ \gamma_{xy}^0$ - components of the neutral surface deformation

$$D_1 = \frac{E_x t^3}{12(1-\mu_{xy}\mu_{yx})}; \quad D_2 = \frac{E_y t^3}{12(1-\mu_{xy}\mu_{yx})}; \quad D_t = \frac{G_{xy}t^3}{12}$$

$$N_1 = -\left(D_1\frac{\partial^3 w}{\partial x^3} + D_3\frac{\partial^3 w}{\partial x\,\partial y^2} + D_1\mu_{xy}\frac{1}{R^2}\frac{\partial w}{\partial x}\right);$$

$$N_2 = -\left(D_2\frac{\partial^3 w}{\partial y^3} + D_3\frac{\partial^3 w}{\partial x^2\,\partial y} + D_2\mu_{yx}\frac{1}{R^2}\frac{\partial w}{\partial y}\right),$$

where $D_3 = D_1\mu_{xy} + 2D_t$ - effective stiffeness

Relation between internal stresses and internal forces-Kirchhoff theory

$$\sigma_x = \frac{T_1}{t} + \frac{12M_1 z}{t^3}; \quad \sigma_y = \frac{T_2}{t} + \frac{12M_2 z}{t^3}; \quad \tau_{xy} = \frac{S}{t} + \frac{12Hz}{t^3}$$

For parabolic distribution of shear stresses through the shell thickness:

$$\tau_{xz} = \frac{6}{5}\frac{N_1}{t}\left[1 - \left(\frac{2z}{t}\right)^2\right]; \quad \tau_{yz} = \frac{6}{5}\frac{N_2}{t}\left[1 - \left(\frac{2z}{t}\right)^2\right]$$

Equations for orthotropic shell in terms of stress function and normal displacement

$$T_1 = t\frac{\partial^2 F}{\partial y^2};\ T_2 = t\frac{\partial^2 F}{\partial x^2} - \frac{1}{R}\left[D_2\left(\frac{\partial^2 w}{\partial y^2} + \frac{w}{R^2}\right) + D_3\frac{\partial^2 w}{\partial x^2}\right];\ S = -t\frac{\partial^2 F}{\partial x\,\partial y}.$$

$$D_1\frac{\partial^4 w}{\partial x^4} + 2D_3\frac{\partial^4 w}{\partial x^2\,\partial y^2} + D_2\frac{\partial^4 w}{\partial y^4} + D_2\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial y^2} + \mu_{yx}\frac{\partial^2 w}{\partial x^2}\right) +$$
$$+\frac{1}{R}\left[\left(D_2\frac{\partial^3 w}{\partial y^3} + D_3\frac{\partial^3 w}{\partial x^2\,\partial y} + D_2\frac{1}{R^2}\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial y} + \left(D_2\frac{\partial^2 w}{\partial y^2} +\right.\right.$$
$$\left.\left.+ D_3\frac{\partial^2 w}{\partial x^2} + D_2\frac{w}{R^2}\right)\left(\frac{1}{R} + \frac{w}{R^2} + \frac{\partial^2 w}{\partial y^2}\right)\right] = p(x,\ y) +$$
$$+ t\left[\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y} + \frac{\partial^2 F}{\partial x^2}\left(\frac{1}{R} + \frac{w}{R^2} + \frac{\partial^2 w}{\partial y^2}\right)\right];$$

$$\frac{1}{E_y}\frac{\partial^4 F}{\partial x^4} + \left(\frac{1}{G_{xy}} - 2\mu_{yx}\frac{1}{E_x}\right)\frac{\partial^4 F}{\partial x^2\,\partial y^2} + \frac{1}{E_x}\frac{\partial^4 F}{\partial y^4} =$$
$$= \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{1}{R}\frac{\partial^2 w}{\partial x^2} +$$
$$+ D_2\frac{1}{E_y R t}\left(\frac{\partial^2}{\partial x^2} - \mu_{xy}\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial y^2} + \frac{w}{R^2}\right) +$$
$$+ D_3\frac{1}{E_y R t}\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2} - \mu_{xy}\frac{\partial^2 w}{\partial y^2}\right).$$

Equations for isotropic shell in terms of stress function and normal displacement

$$E_x = E_y = E, \quad \mu_{xy} = \mu_{yx} = \mu, \quad G_{xy} = E/[2(1+\mu)]$$

$$D\nabla^2\nabla^2 w + D\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right) + D\frac{1}{R}\left[\frac{\partial w}{\partial y}\frac{\partial}{\partial y}\left(\nabla^2 w + \frac{w}{R^2}\right) + \right.$$
$$\left. + \left(\frac{1}{R} + \frac{w}{R^2} + \frac{\partial^2 w}{\partial y^2}\right)\left(\nabla^2 w + \frac{w}{R^2}\right)\right] = p + t\left[\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 F}{\partial x^2} - \right.$$
$$\left. - 2\frac{\partial^2 F}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y} + \frac{\partial^2 F}{\partial x^2}\left(\frac{1}{R} + \frac{w}{R^2} + \frac{\partial^2 w}{\partial y^2}\right)\right];$$

$$\nabla^2\nabla^2 F = E\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{1}{R}\frac{\partial^2 w}{\partial x^2}\right] +$$
$$+ D\frac{1}{Rt}\left(\frac{\partial^2}{\partial x^2} - \mu\frac{\partial^2}{\partial y^2}\right)\left(\nabla^2 w + \frac{w}{R^2}\right); \qquad D = \frac{Et^3}{12(1-\mu^2)}$$

Simplified equations for orthotropic shell in terms of stress function and normal displacement

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} = p + h\left[\frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - \right.$$
$$\left. -2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2}\left(\frac{\partial^2 w}{\partial y^2} + \frac{w}{R^2} + \frac{1}{R}\right)\right];$$

$$\frac{1}{E_y} \frac{\partial^4 F}{\partial x^4} + \left(\frac{1}{G_{xy}} - 2\mu_{yx} \frac{1}{E_x}\right) \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{1}{E_x} \frac{\partial^4 F}{\partial y^4} =$$
$$= \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 w}{\partial y^2} + \frac{1}{R}\right).$$

$$T_1 = h \frac{\partial^2 F}{\partial y^2}, \quad T_2 = h \frac{\partial^2 F}{\partial y^2}, \quad S = -h \frac{\partial^2 F}{\partial x \partial y}.$$

Simplified equations for isotropic shell in terms of stress function and normal displacement

$$D\nabla^2\nabla^2 w = p + h\left[\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\,\frac{\partial^2 F}{\partial x\,\partial y}\,\frac{\partial^2 w}{\partial x\,\partial y} + \frac{\partial^2 F}{\partial x^2}\left(\frac{\partial^2 w}{\partial y^2} + \frac{w}{R^2} + \frac{1}{R}\right)\right];$$

$$\nabla^2\nabla^2 F = E\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\left(\frac{1}{R} + \frac{\partial^2 w}{\partial y^2}\right)\right].$$

$$T_1 = h\,\frac{\partial^2 F}{\partial y^2}, \quad T_2 = h\,\frac{\partial^2 F}{\partial y^2}, \quad S = -\,h\,\frac{\partial^2 F}{\partial x\,\partial y}\,.$$

Type of stresses in shell

1. **Mixed state-stresses are produced by bending and membrane forces**
2. **Pure bending state-influence of membrane stresses is negligible**
3. **Membrane state-influence of bending stresses is negligible**

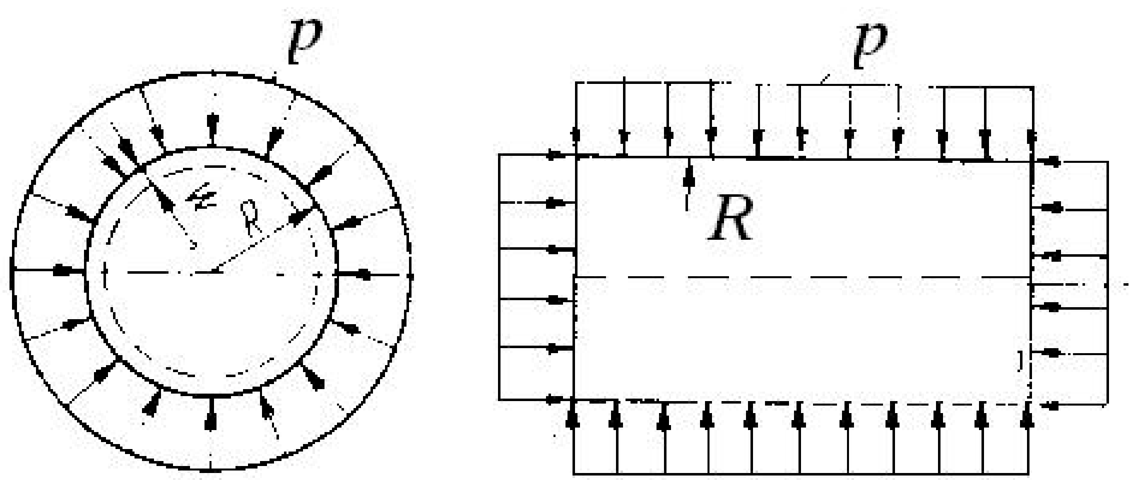
$$M_1 = M_2 = H = 0,$$

$$N_1 = N_2 = 0, \quad \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} + \frac{w}{R^2} = 0.$$

Equilibrium equations in terms of forces

$$\frac{\partial T_1}{\partial x} + \frac{\partial S}{\partial y} = 0; \quad \frac{\partial S}{\partial x} + \frac{\partial T_2}{\partial y} = 0; \quad T_2 = -pR.$$

Axisymmetric bending of cylindrical shell under hydrostatic pressure-formulation



$$Dw^{\mathrm{IV}}(x) = p + t\left(\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{1}{R}\frac{\partial^2 F}{\partial x^2}\right)$$

$$\frac{\partial^4 F}{\partial x^4} = -\frac{E}{R}\frac{\partial^2 w}{\partial x^2}.$$

Bending of cylindrical shell under hydrostatic pressure-solution

$$T_1 = t\,\frac{\partial^2 F}{\partial y^2}; \quad T_2 = t\,\frac{\partial^2 F}{\partial x^2}; \quad S = -t\,\frac{\partial^3 F}{\partial x\,\partial y};$$

$$M_1 = -Dw''; \quad M_2 = -\mu Dw''; \quad N_1 = -Dw'''; \quad N_2 = H = 0.$$

Differential equation in terms of membrane forces

$$Dw^{\mathrm{IV}}(x) = p + T_1 w''(x) + T_2/R.$$

$$T_1 \cong -p\pi R^2/(2\pi R) = -pR/2; \quad T_2 = -(Etw/R + \mu pR/2).$$

Differential equation in terms of displacements

$$Dw^{\mathrm{IV}}(x) + pRw''(x)/2 + Etw(x)/R^2 = (1 - \mu/2)\,p.$$

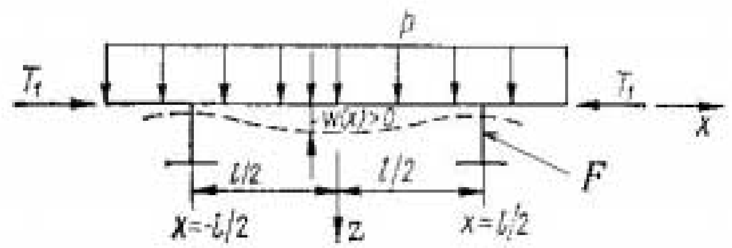
Solution

$$w(x) = [pR^2/(Et)](1 - \mu/2) + C_1 \operatorname{ch}\delta x \cos\gamma x + C_2 \operatorname{sh}\delta x \sin\gamma x + C_3 \operatorname{ch}\delta x \sin\gamma x + C_4 \operatorname{sh}\delta x \cos\gamma x.$$

where $\delta = \alpha\sqrt{1-\beta}; \quad \gamma = \alpha\sqrt{1+\beta},$

$$\alpha = \sqrt[4]{3(1-\mu^2)/(R^2t^2)}; \quad \beta = (R/t)^2\, p\,\sqrt{3(1-\mu^2)}\,/(2E).$$

Bending of cylindrical shell between stiffeners



Symmetry: $C_3 = C_4 = 0.$

$$w(x) = [pR^2/(Et)](1-\mu/2) + C_1 \operatorname{ch} \delta x \cos \gamma x + C_2 \operatorname{sh} \delta x \sin \gamma x.$$

B.C.: $x = \pm \frac{l}{2} \quad \frac{dw}{dx} = 0; \quad x = l/2 \quad w = 2DR^2 w'''/(EF).$

$$C_1 = -\frac{2pR^2}{Et}\left(1-\frac{\mu}{2}\right)\frac{u_1 \operatorname{ch} u_1 \sin u_2 + u_2 \operatorname{sh} u_1 \cos u_2}{u_2 \operatorname{sh} 2u_1 + u_1 \sin 2u_2}\varepsilon_1;$$

$$C_2 = -\frac{2pR^2}{Et}\left(1-\frac{\mu}{2}\right)\frac{u_2 \operatorname{ch} u_1 \sin u_2 - u_1 \operatorname{sh} u_1 \cos u_2}{u_2 \operatorname{sh} 2u_1 + u_1 \sin 2u_2}\varepsilon_1.$$

where

$$\varepsilon_1 = \frac{1}{1 + ltF_1(u_1, u_2)/F}; \quad u_1 = \frac{\delta l}{2} = u\sqrt{1+\beta}; \quad u_2 = \frac{\gamma l}{2} = u\sqrt{1-\beta}$$

$$F_1(u_1, u_2) = \sqrt{1-\beta^2}\,\frac{\operatorname{ch} 2u_1 - \cos 2u_2}{u_2 \operatorname{sh} 2u_1 + u_1 \sin 2u_2}; \quad u = 0{,}6425\frac{l}{\sqrt{Rt}}.$$

Bending of cylindrical shell between stiffeners- Sanden and Gunther solution

Midbay

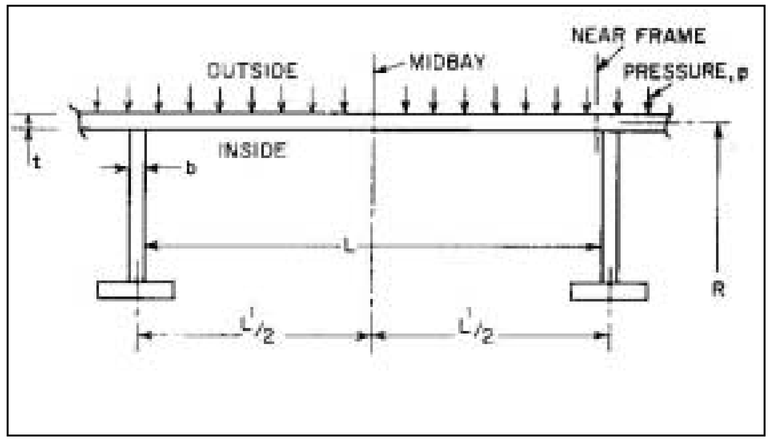
$$\sigma_y = \frac{-pR}{t}\left[1 + \Gamma(H_M \pm \mu H_E)\right]$$

$$\sigma_x = \frac{-pR}{t}\left[\tfrac{1}{2} \pm \Gamma H_K\right]$$

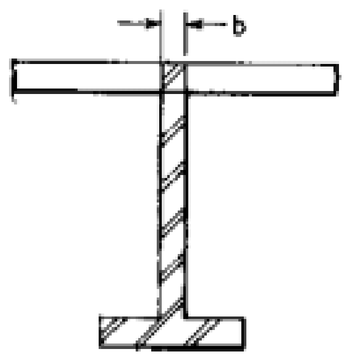
Near to the frame

$$\sigma_y = \frac{-pR}{t} \times \left\{1 - \Gamma\left[1 \pm \mu\left(\frac{3}{1-\mu^2}\right)^{1/2} K\right]\right.$$

$$\sigma_x = \frac{-pR}{t}\left[\tfrac{1}{2} \mp \Gamma\left(\frac{3}{1-\mu^2}\right)^{1/2} K\right]$$



Effective frame area



Bending of cylindrical shell between stiffeners-Sanden and Gunther solution (cont)

in which

$\frac{-pR}{t}$ = hoop stress, psi

R = radius to midplane of shell, in.
p = external pressure, psi
Γ = frame deflection parameter

$$= \frac{(1 - \mu/2) - B}{1 + \beta} = \frac{0.85 - B}{1 + \beta} \quad \text{for} \quad \mu = 0.3$$

B = ratio of shell area under the frame faying flange, to total frame area plus shell area under frame faying flange.
β = degree of flexibility provided by frame (for a very large area β will approach zero)
θ = a slenderness parameter dependent on shell thickness, frame spacing, and radius of curvature of shell plating
N, H, K = transcendental functions that define bending effect on shell due to local framing, where
N = effect on deflection of frame
H = bending effect in shell reflected at midspan
K = bending effect in shell reflected near frame

$$B = \frac{bt}{A + bt}$$

where b = width of faying flange, in.
t = shell thickness, in.
A = frame area, sq in.

$$\beta = \frac{2N}{A + bt}\left(\frac{1}{3(1 - \mu^2)}\right)^{1/4}(Rt^3)^{1/2}$$
$$= \frac{1.555N(Rt^3)^{1/2}}{A + bt} \quad \text{for} \quad \mu = 0.3$$

where

$$N = \frac{\cosh\theta - \cos\theta}{\sinh\theta + \sin\theta}$$

$$\theta = L\left[\frac{3(1 - \mu^2)}{R^2t^2}\right]^{1/4} = \frac{1.285L}{(Rt)^{1/2}} \quad \text{for} \quad \mu = 0.3$$

$$H_M = -2\left[\frac{\sinh\theta/2\cos\theta/2 + \cosh\theta/2\sin\theta/2}{\sinh\theta + \sin\theta}\right]$$

$$H_K = -2\left(\frac{3}{1 - \mu^2}\right)^{1/2} \times\left[\frac{\sinh\theta/2\cos\theta/2 - \cosh\theta/2\sin\theta/2}{\sinh\theta + \sin\theta}\right]$$

$$K = \frac{\sinh\theta - \sin\theta}{\sinh\theta + \sin\theta}$$

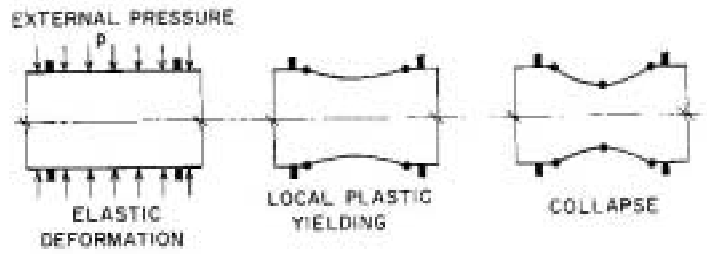
Sanden and Gunther solution-expression of critical pressure resulting in yield

$$p = \frac{\dfrac{2\sigma_y t}{D}}{0.5 + 1.815K\left(\dfrac{0.85 - B}{1+\beta}\right)}$$

(yielding at frame)

$$p = \frac{\dfrac{2\sigma_y t}{D}}{1 + H\left(\dfrac{0.85 - B}{1+\beta}\right)}$$

(yielding at midbay)



where

σ_y = yield stress, psi

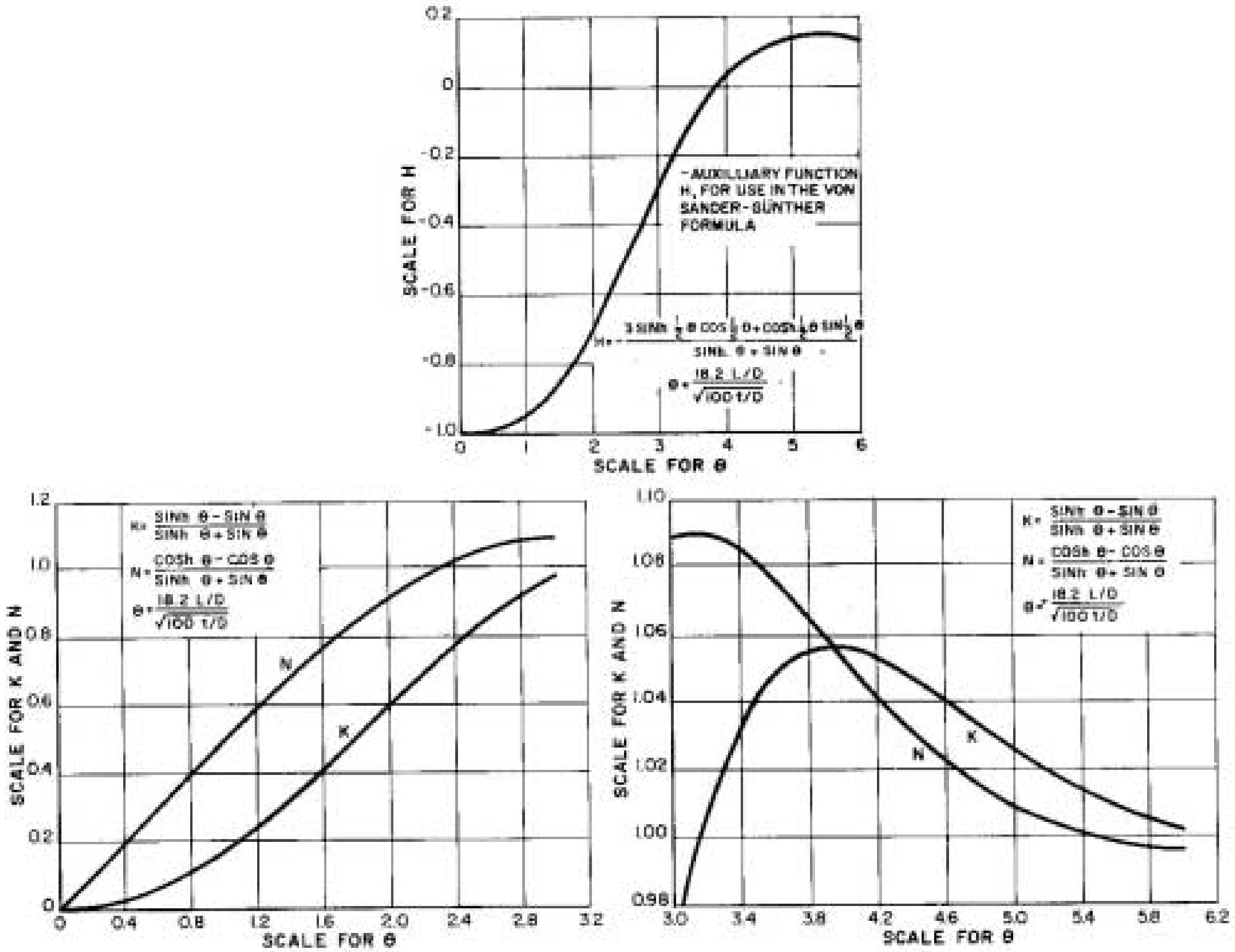
$$H = H_M + \mu H_E$$

$$= -\frac{2\left[1 + \left(\dfrac{3\mu^2}{1-\mu^2}\right)^{1/2}\right]\sinh\dfrac{\theta}{2}\cos\dfrac{\theta}{2} - 2\left[1 - \left(\dfrac{3\mu^2}{1-\mu^2}\right)^{1/2}\right]\cosh\dfrac{\theta}{2}\sin\dfrac{\theta}{2}}{\sinh\theta + \sin\theta}$$

$$= -\frac{3.089\sinh\dfrac{\theta}{2}\cos\dfrac{\theta}{2} + 0.917\cosh\dfrac{\theta}{2}\sin\dfrac{\theta}{2}}{\sinh\theta + \sin\theta}$$

for $\mu = 0.3$

Sanden and Gunther solution-charts for functions N,K H



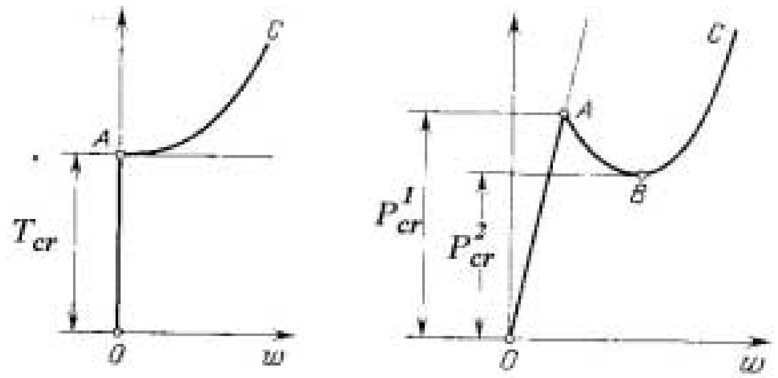
Stability of shell-formulation

Variations of equilibrium position

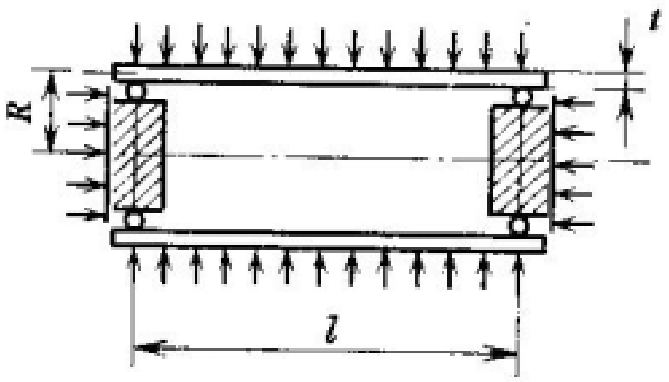
$$\tilde{w}(x,\ y) = w_2(x,\ y) - w_1(x,\ y); \quad \widetilde{F}(x,\ y) = F_2(x,\ y) - F_1(x,\ y)$$

Differential equation of stability of the shell

$$D\nabla^2\nabla^2\tilde{w} = t\left[\frac{\partial^2 F_1}{\partial y^2}\frac{\partial^2 \tilde{w}}{\partial x^2} + \frac{\partial^2 \widetilde{F}}{\partial y^2}\frac{\partial^2 w_1}{\partial x^2} - 2\left(\frac{\partial^2 F_1}{\partial x\,\partial y}\frac{\partial^2 \tilde{w}}{\partial x\,\partial y} + \right.\right.$$
$$\left. + \frac{\partial^2 \widetilde{F}}{\partial x\,\partial y}\frac{\partial^2 w_1}{\partial x\,\partial y}\right) + \frac{\partial^2 F_1}{\partial x^2}\left(\frac{\tilde{w}}{R^2} + \frac{\partial^2 \tilde{w}}{\partial y^2}\right) +$$
$$\left. + \frac{\partial^2 \widetilde{F}}{\partial x^2}\left(\frac{1}{R} + \frac{w_1}{R^2} + \frac{\partial^2 w_1}{\partial y^2}\right)\right];$$
$$\nabla^2\nabla^2\widetilde{F} = E\left(2\frac{\partial^2 w_1}{\partial x\,\partial y}\frac{\partial^2 \tilde{w}}{\partial x\,\partial y} - \frac{\partial^2 w_1}{\partial x^2}\frac{\partial^2 \tilde{w}}{\partial y^2} - \frac{\partial^2 \tilde{w}}{\partial x^2}\frac{\partial w_1}{\partial y^2} - \frac{1}{R}\frac{\partial^2 \tilde{w}}{\partial x^2}\right).$$



Stability of the simple supported shell



Relations for the membrane state of the shell

$$(T_1)_1 = -pR/2; \quad (T_2)_1 = -pR; \quad (S)_1 = 0, \quad w_1(x) = \text{const.}$$

$$\frac{\partial^2 F_1}{\partial y^2} = -\frac{pR}{2t}; \quad \frac{\partial^2 F_1}{\partial x^2} = -\frac{pR}{t}; \quad \frac{\partial^2 F_1}{\partial x\, \partial y} = 0.$$

$$D\,\nabla^2\nabla^2\tilde{w} = -\frac{pR}{2}\frac{\partial^2\tilde{w}}{\partial x^2} - pR\left(\frac{\tilde{w}}{R^2} + \frac{\partial^2\tilde{w}}{\partial y^2}\right) + \frac{\partial^2\widetilde{F}}{\partial x^2}\left(\frac{t}{R} + \frac{tw_1}{R^2}\right);$$

$$\nabla^2\nabla^2\widetilde{F} = -\frac{E}{R}\frac{\partial^2\tilde{w}}{\partial x^2}.$$

Stability equation

$$\nabla^2\nabla^2\left[D\nabla^2\nabla^2\tilde{w} + \frac{pR}{2}\frac{\partial^2\tilde{w}}{\partial x^2} + pR\left(\frac{\tilde{w}}{R^2} + \frac{\partial^2\tilde{w}}{\partial y^2}\right)\right] + \frac{tE}{R^2}\frac{\partial^4\tilde{w}}{\partial x^4} = 0.$$

Stability of simple supported shell-Mises formula

$$\tilde{w}(x,\ y)=a_{mn}\sin m\pi x/l\sin n\pi y/R\quad\left(\textbf{B.C.}\ \ \tilde{w}=\frac{\partial^2\tilde{w}}{\partial x^2}=0\right)$$

$$p_e=(p_{mn})_{\min}=\min\left\{\frac{Et}{R}\,\frac{1}{m^2\pi^2R^2/(2l^2)+n^2-1}\times\right.$$
$$\left.\times\left[\frac{m^4\pi^4R^4/l^2}{(m^2\pi^2R^2/l^2+n^2)^2}+\frac{D}{EtR^2}(m^2\pi^2R^2/l^2+n^2)^2\right]\right\}.$$

Mises formula $m=1$

$$p_e=\min\left\{\frac{Et}{R}\,\frac{1}{0{,}5\alpha^2+n^2-1}\left[\frac{\alpha^4}{(\alpha^2+n^2)^2}+\frac{t^2}{12\,(1-\mu^2)\,R^2}(\alpha^2+n^2)\right]\right\}.$$

where $\alpha=\pi R/l$

Stability of simple supported shell-David Taylor model basin formula

Assumption

$$0{,}5\alpha^2 + n^2 - 1 \approx 0{,}5\alpha^2 + n^2.$$

$$P_{cr} = \frac{2.42E \quad (t/2R)^{5/2}}{(1-\nu^2)^{0.75} \; [(L/2R) - 0.447 \; (t/2R)^{1/2}]}$$

where E = Young's modulus of elasticity
ν = Poisson's ratio
t = wall thickness of cylinder
R = mean shell radius
L = length between adjacent stiffeners

The model takes into account the terms ~ N_2/R

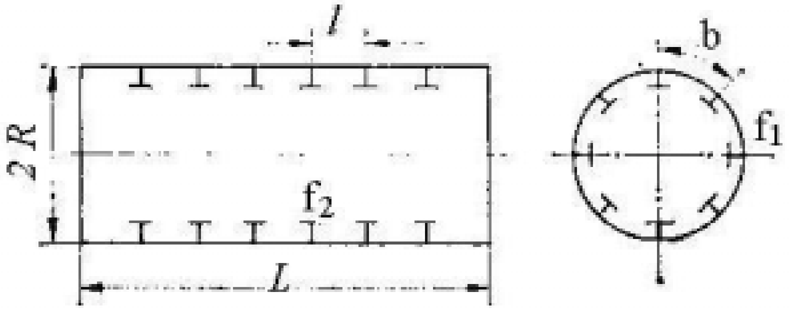
$$p_e = \min\left[\frac{Et}{R}\frac{1}{0{,}5\alpha^2 + n^2 - 1}\left\{\frac{t^2}{12\,(1-\mu^2)\,R^2}\left[(\alpha^2 + n^2 - 1)^2 + (1-\mu)\,\alpha^2\right] + \right.\right.$$
$$\left.\left. + \left(\frac{\alpha^2}{\alpha^2 + n^2}\right)^2\left[\alpha^2 + \frac{t^2}{12\,(1-\mu^2)\,R^2}\,(\alpha^2 - \mu n^2)\,(\alpha^2 + n^2 - 1)\right]\right\}\right].$$

where $\alpha = \pi R/l$

Infinitely long shell $\alpha = 0$

$$p_e = \min\,[D\,(n^2 - 1]/R^3] = 3D/R^3,$$

General instability of reinforced shell-orthotropic shell model



$$p_e = \min\{[D_n/R^3 + t\beta_n\alpha^2/(Rt_n)]/\gamma_n\}.$$

$$D_n = D_1\alpha^4 + 2D_3\alpha^2(n^2-1) + D_2(n^2-1)^2 + (D_3 - \mu_{yx}D_2)\alpha^2;$$

$$\beta_n = \alpha^2 + (\alpha^2 - \mu_{xy}n^2)[D_3\alpha^2 + D^2(n^2-1)]/(E_yR^2t);\ \ \alpha = \pi R/L;$$

$$t_n = \alpha^4\delta_2 + 2\delta_3\alpha^2n^2 + \delta_1 n^4;\ \ \gamma_n = 0{,}5\alpha^2 + n^2 - 1,$$

$$\delta_1 = 1/E_x;\ \ \delta_2 = 1/E_y;\ \ \delta_3 = 1/(2G_{xy}) - \mu_{yx}/E_x.$$

$$E_xt = E(t + f_1/b);\ \ D_1 = EI_1/b + D;\ \ E_yt = E(t + f_2/l);\ \ D_2 = EI_2/l + D.$$

$$E_x = E[1 + f_1/(bt)];\ E_y = E[1 + f_2/(lt)];\ D_1 = D + EI_1/b;\ D_2 = D + EI_2/l$$

$$D_t = Gt^3/12 = (1-\mu)D/2.$$

$$\mu_{xy} = \mu E/E_x;\quad \mu_{yx} = \mu E/E_y;\quad G = E/[(2(1+\mu)]$$

Solution for the shell with circular frames and no stringers

$$E_x = E; \quad E_y = E\,[1 + I_2/(lt)]; \quad D_1 = D; \quad D_2 = EI_2/l + D;$$
$$\mu_{xy} = \mu; \quad \mu_{yx} = \mu\, E/E_y; \quad D_3 = 2D_t + \mu_{xy} D_1 = D;$$
$$\delta_1 = 1/E_x; \quad \delta_2 = 1/E_y; \quad \delta_3 = (1 + \mu - \mu E/E_y)/E.$$

$$p_e = \min \frac{1}{0{,}5\alpha^2 + n^2 - 1} \left\{ \frac{EI_2}{R^3 l} \left[(n^2 - 1)^2 + \frac{\alpha^2}{\Delta_n} (n^2 - 1)(\alpha^2 - \mu n^2) \right] + \right.$$
$$+ \frac{Et}{R} \frac{(1 + k)\,\alpha^4}{\Delta_n} + \frac{D}{R^3} \left[\frac{\alpha^2}{\Delta_n} (\alpha^2 - \mu n^2)(\alpha^2 + n^2 - 1) + \right.$$
$$\left. \left. + (1 - \mu)\,\alpha^2 + (\alpha^2 + n^2 - 1)^2 \right] \right\},$$

$$k = I_2/(lt); \quad \Delta_n = (\alpha^2 + n^2)^2 + kn^2\,[2\,(1 + \mu)\,\alpha^2 + n^2].$$

For $EI_2 >> Dl, \ k << 1$ **one can put:** $k = 0, \ D = 0$

$$p_e = \min \left\{ \frac{E}{0{,}5\alpha^2 + n^2 - 1} \left[\frac{I_2}{R^3 l} (n^2 - 1)^2 + \frac{t}{R} \frac{\alpha^4}{(\alpha^2 + n^2)^2} \right] \right\}.$$

General instability-Kendrick-Bryant formula

$$p_e = \frac{Et}{R}\frac{\alpha^4}{\left(n^2 - 1 + \frac{\alpha^2}{2}\right)(n^2+\alpha^2)^2} + \frac{(n^2-1)EI}{R^3L_s}$$

where

$\alpha = \dfrac{\pi R}{L_s}$

L_s = length between bulkheads

n = number of circumferential lobes

I = inertia of frame including one frame space of shell plating

R = mean shell radius

"Thumb" rules for the preliminary thickness and frame spacing selection

$$t \cong \beta \frac{PcR}{\sigma y}$$

t = thickness of shell
Pc = collapse pressure = pressure at maximum operating depth times factor of safety
R = mean radius of shell
σy = yield strength of material
β = frame reduction factor

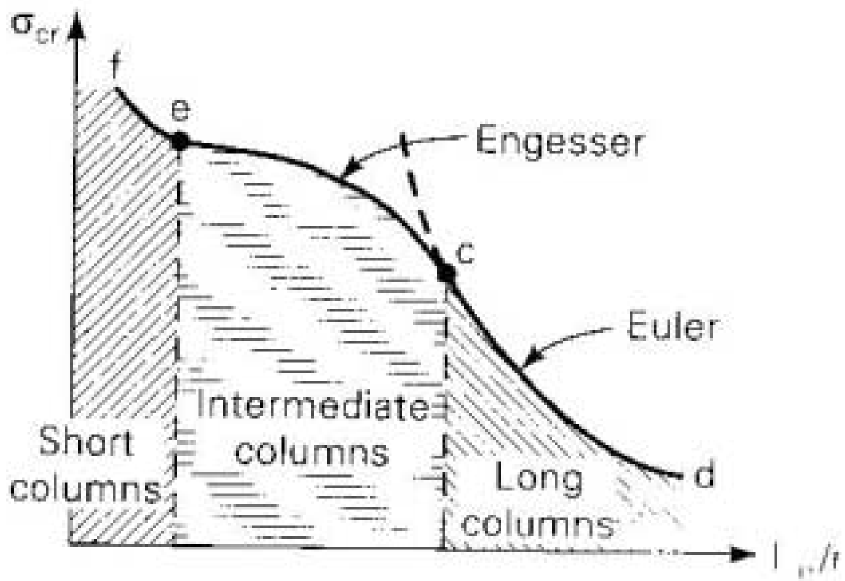
L_f = frame spacing A_f = sectional area of frame

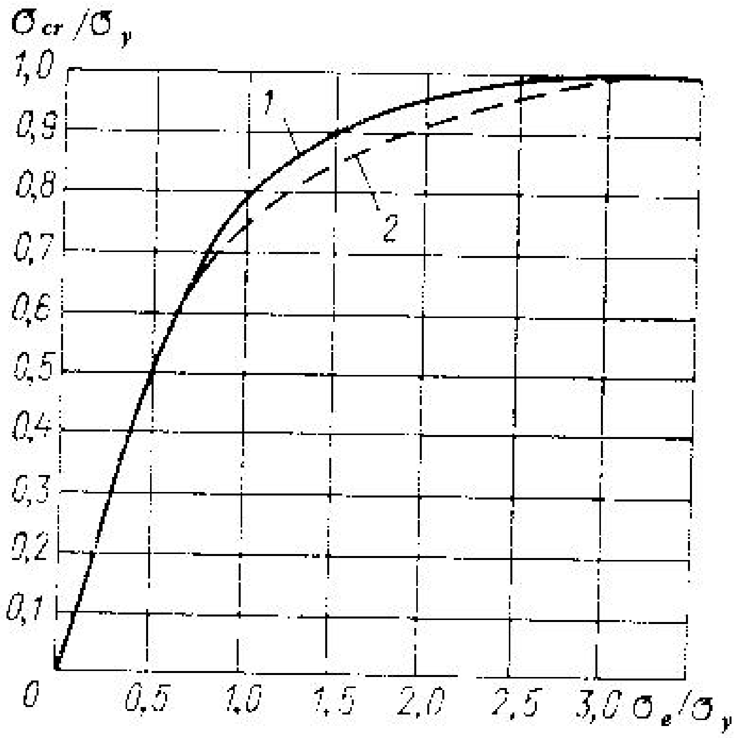
$L_f \cong \sqrt{R\,t}$ $A_f \cong 1/2\, L_f\, t$

Euler force and influence of plasticity

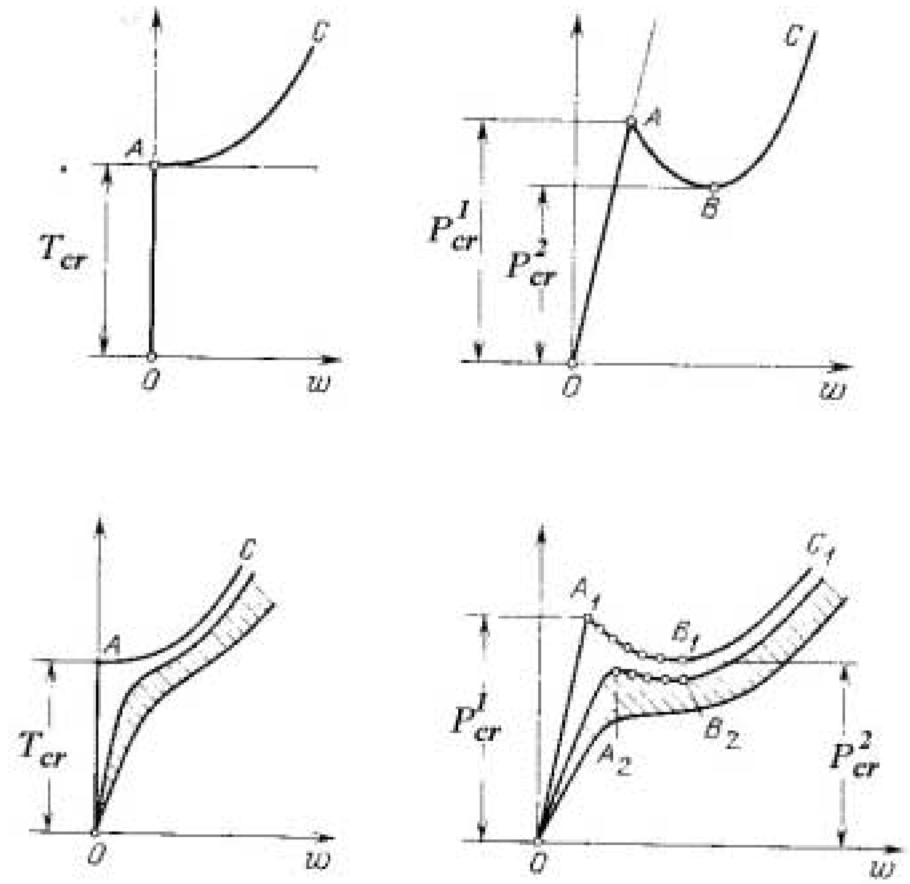
$$T_e = \frac{\pi^2 E\, I}{(\mu\, l)^2};$$

$$T_e = \frac{\pi^2 E_t\, I}{(\mu\, l)^2}; \quad E_t - \text{tangent modulus}$$

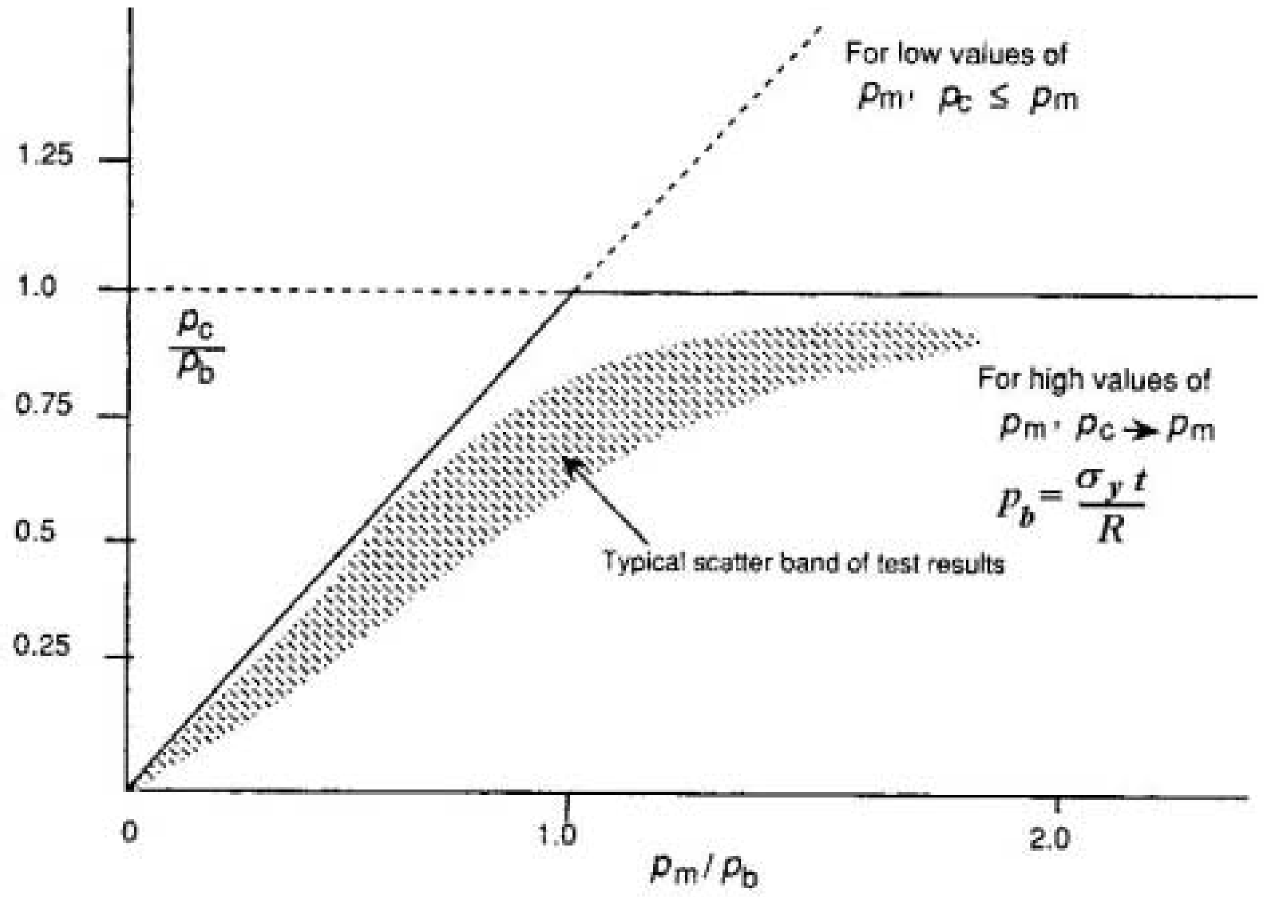




Influence of the inelastic behavior and imperfections on the critical pressure

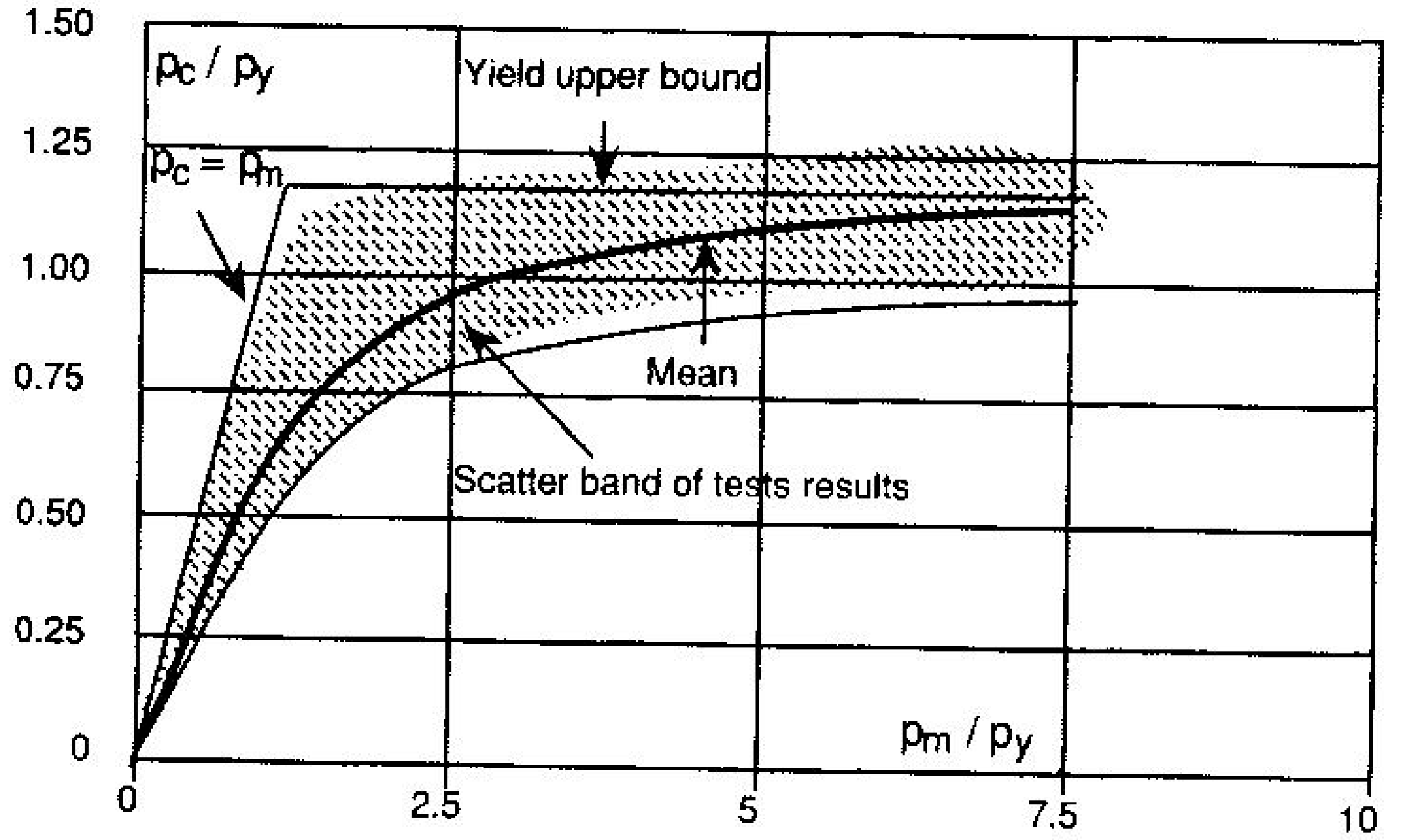


Interframe shell collapse results-empirical data related to the "boiler" stress



Empirical plot of model pressure test results (**Burcher and Rydill**)

Interframe shell collapse results-empirical data related to the shell yield stress



Interframe shell collapse results (**Burcher and Rydill**)

Thinness ratio-shell stability

Shell instability

$\lambda = [\,(L/2R)^2 / (t/2R)^3]^{0.25} \sqrt{(\sigma_{yp} / E)}$ = Windenburg thinness ratio 'λ',
L = unsupported length of the shell between adjacent frames
t = wall thickness
R = mean shell radius
σ_{yp} = yield stress
E = Young's modulus of elasticity

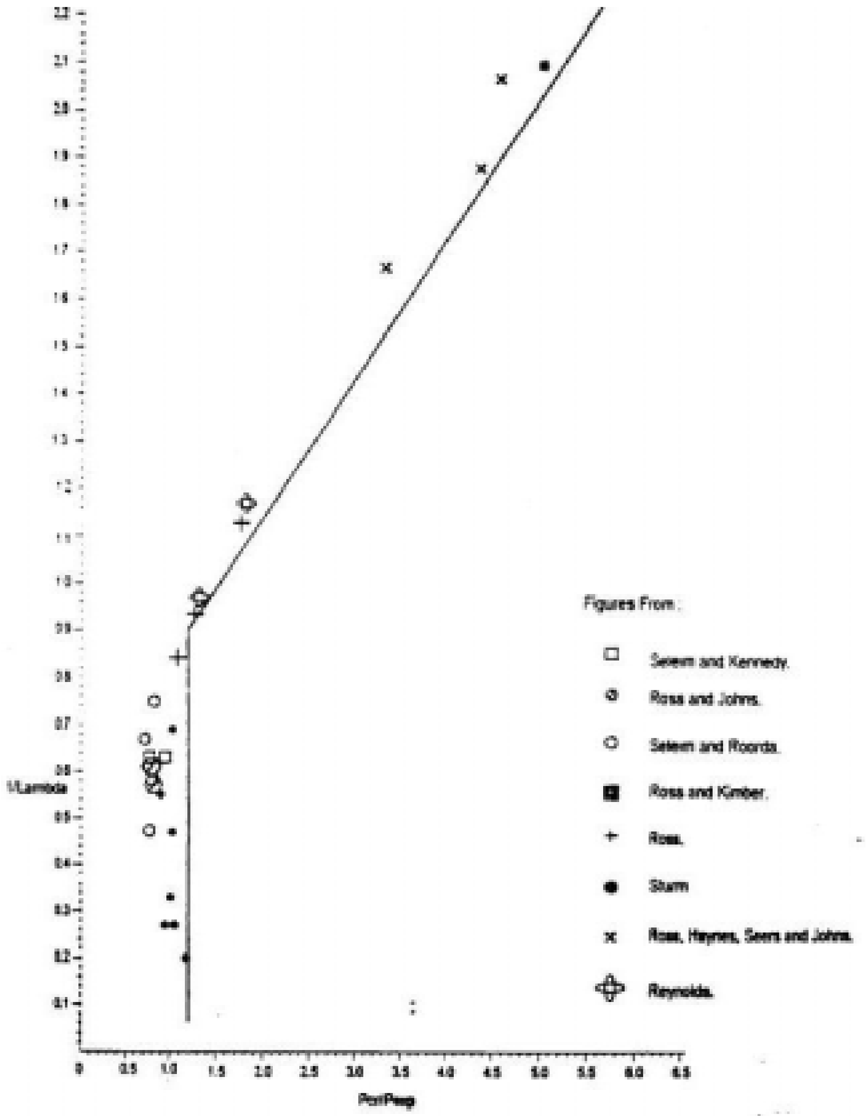
General instability

$\lambda^1 = [L_b/D_f)^2/(t^1/D_f)^3]^{0.25} \sqrt{(\sigma_{yp}/E)}$
L_b = length between bulkheads
D_f = diameter of the centroid of the ring-shell combination
t^1 = wall thickness of an unstiffened vessel equivalent to the ring-stiffened case

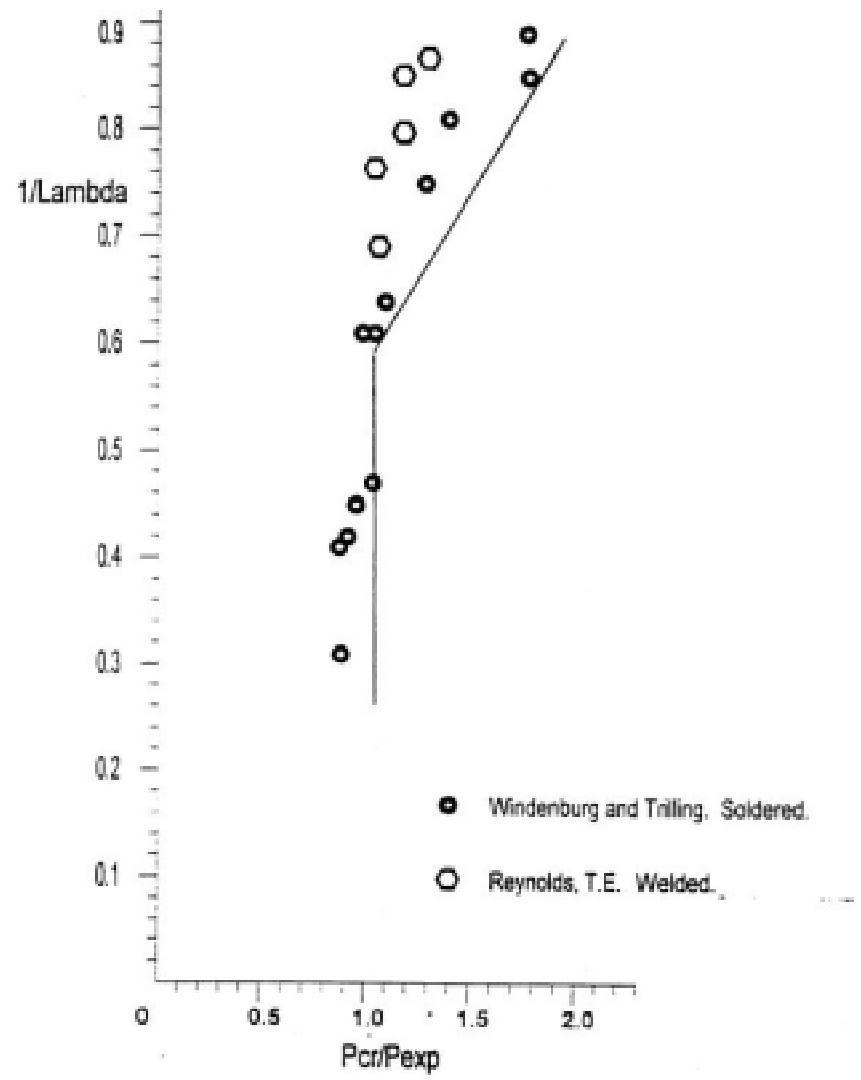
For columns we had:

$$\lambda = \frac{L}{r}; \qquad r = \sqrt{I/F} \text{ – radius of giration}$$

Shell instability-design charts



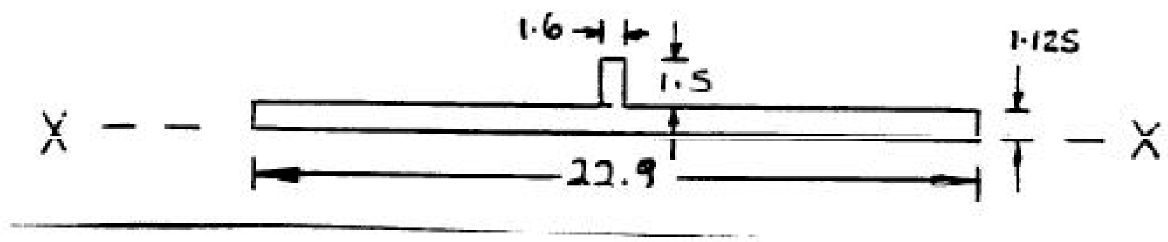
Design chart for the shell instability of machined circular cylinders



Thinness ratio-example of calculation

Sample Calculation for λ^1 for model

A typical ring-shell combination is shown in Figure 23, where the dimensions are in mm



Internal diameter = 101.6mm

Figure 23 Typical ring-shell combination for model P4

L_b = 217mm
σ_{yp} = 200 MPA
E = 200 GPa

$\sqrt{(\sigma_{yp}/E)} = \sqrt{(200/200 \times 10^3)} = 0.0316$

A = cross-sectional area of a typical ring-shell combination

= 28.163 mm^2

L = Unsupported length between adjacent frames = 22.9 mm

Thinness ratio-example of calculation (cont)

Ay = first moment of area of the ring-shell combination about XX

= $22.9 \times 1.125 \times 0.5625 + 1.6 \times 1.5 \times 1.875$

= $18.991\ mm^3$

$\bar{y}$ = distance of centroid of the ring-shell combination from XX

= Ay/A

$\underline{\bar{y} = 0.674\ mm}$

D_f = diameter of the centroid of the ring shell combination

= $101.6 + 2\bar{y}$

$\underline{D_f = 102.95\ mm}$

I_{xx} = 2nd moment of area of ring-shell combination about XX

= $22.9 \times 1.125^3/3 + 1.6 \times 1.5^3/12 + 1.6 \times 1.5 \times 1.875^2$

Thinness ratio-example of calculation (end)

$\underline{I_{xx} = 19.756\ mm^4}$

I_{NA} = 2nd moment of area of the ring-shell combination about its own neutral axis

= $I_{xx} - \bar{y}^2 \times A$

= $19.756 - 0.674^2 \times 28.163$

$\underline{I_{NA} = 6.962}$

Let t^1 = thickness of an equivalent unstiffened cylinder, so that,

$$\frac{22.9 \quad x \quad t^{1^3}}{12} = I_{NA}$$

$\therefore \quad \underline{t^1 = 1.539\ mm}$

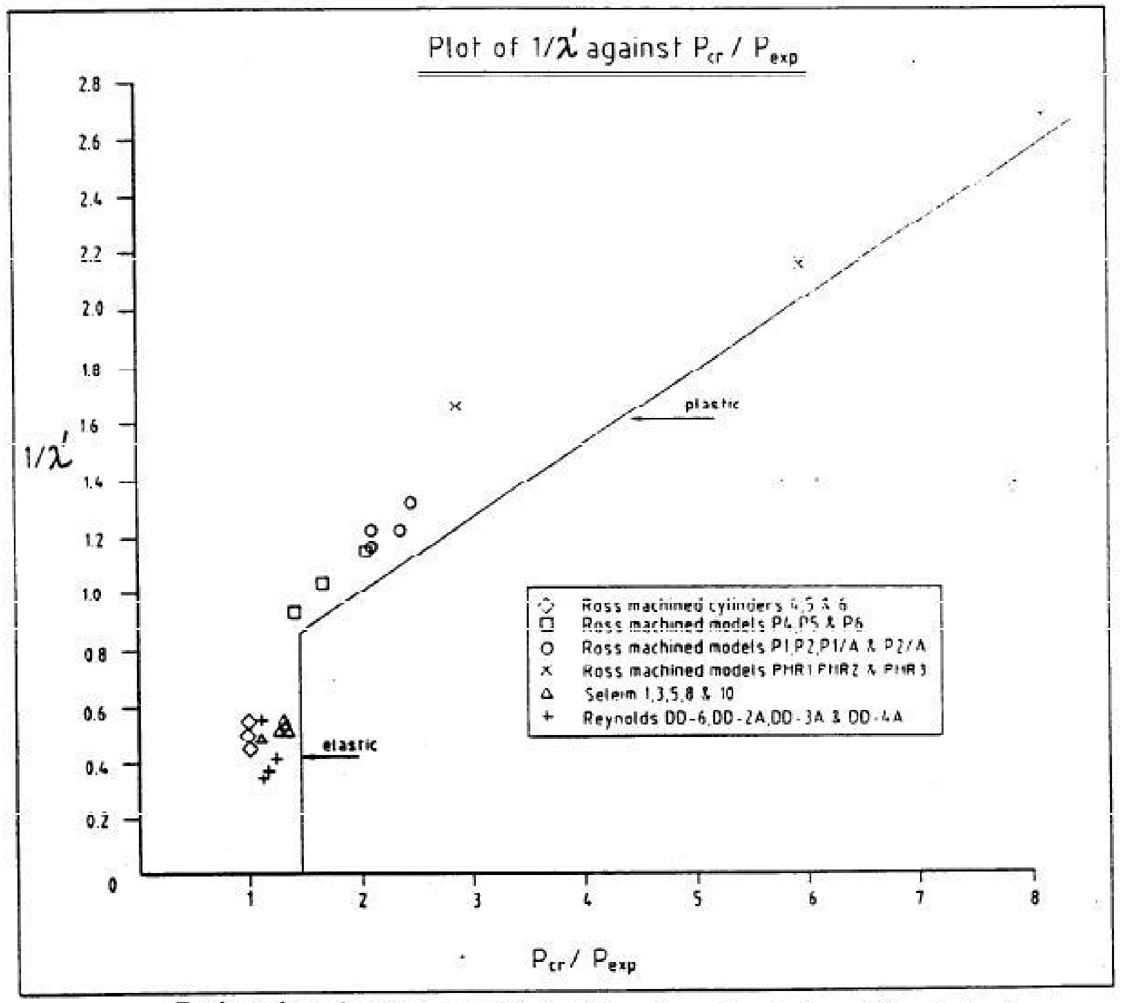
$\lambda^1 = [L_b/D_f)^2 / t^1 /D_f)^3\ ]^{0.25} \sqrt{}\ (\sigma_{yr}/E)$

= $[(217/102.95)^2/(1.539/102.95)^3]^{0.25} \times 0.0316$

$\underline{\lambda^1 = 1.074}$

$\underline{1/\lambda^1 = 0.931}$

Design chart for general instability



Design chart for the general instability of machined ring-stiffened circular cylinders The Pcr calculated using FEM or by Bryant/Kendrick formula

Spherical shells

$$P_1 = \frac{2E\left(\frac{h}{R}\right)^2}{[3(1-\mu^2)]^{1/2}}$$

P_1 = elastic buckling pressure ; R = mean radius of shell ;
E = modulus of elasticity ; μ = Poisson's ratio .
h = shell thickness ;

$$P_1 = 1.21\,E\left(\frac{h}{R}\right)^2 \qquad (\mu = 0.3)$$

Spherical shell-influence of inelasticity and imperfections

David Taylor model basin formulas

$$P_3 = 0.84\ E\left(\frac{h}{R_0}\right)^2$$

P_3 = empirical buckling pressure for near perfect sphere
R_0 = outer shell radius

$$P_E = 0.84\ (E_s E_t)^{1/2}\left(\frac{h}{R_0}\right)^2 \text{ for } \mu = 0.3$$

where

P_E = inelastic collapse pressure of near-perfect sphere
E_s = secant modulus
E_t = tangent modulus; and change in Poisson's ratio from elastic to plastic range is neglected, a conservative assumption

Appendix I-Pressure hull materials

Material	σ_{UTS}(MPa)	E (GPa)	ρ kg/m	σ_{UTS}/ρ m/s	E/ρ $(m/s)^2$
Medium strength mild steel	370	200	7840	47×10^3	26×10^6
High-strength mild steel	1550	200	7840	198×10^3	26×10^6
Aluminium alloy	430	70	2800	154×10^3	25×10^6
Titanium alloy	690	120	4500	153×10^3	27×10^6
High-strength GFRP	1600	60	2000	800×10^3	30×10^6
CFRP	1400	170	1600	875×10^3	105×10^6
Boron composite	1300	270	2000	650×10^3	135×10^6
Piano wire	3000	200	7840	380×10^3	26×10^6