POWER AND BANDWIDTH EFFICIENT ROUTING CONTROL FOR COMMUNICATING AUTONOMOUS UNDERWATER VEHICLES

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Abstract— **This paper presents a procedure for constructing controllers that limit and route information flow in a network of Autonomous Underwater Vehicles (AUVs). Particular attention is paid to the objectives of maximizing throughput and bandwidth efficiency while minimizing power consumption. A receding horizon control approach is adopted and shown to yield piecewise continuous optimal controllers that are unique. Simulation results for the receding horizon system are also presented.**

I. Introduction

Underwater vehicles are being increasingly employed in exploring the world's oceans. Many missions, such as studying ocean processes, weather forecasting, and mapping the ocean floor, need a small, inexpensive craft to take measurements. Current missions require a ship to support one or two underwater vehicles, making widespread underwater exploration prohibitively expensive. Efforts are underway to network multiple autonomous underwater vehicles (AUVs) and make them independent of support vessels, thereby reducing cost and making underwater exploration more accessible.

Researchers have acknowledged the need for AUVs working in concert to provide more effective sampling of the ocean [16]. It is envisioned that vehicles working in formation will provide effective sampling of the ocean. An Autonomous Ocean Sampling Network (AOSN) of underwater vehicles and support hardware has been proposed in [15]. Among the work to make such a network possible are advances in acoustic signal processing [10], [9]. Sophisticated signal processing is necessary to combat the Doppler and multipath effects, fading, absorbtion, and ambient noise that occur in underwater acoustic channels [5].

There are three distinguishing characteristics about the AUV communication networks considered in this paper. First, the ocean is a broadcast environment with very limited bandwidth, so some form of frequency reuse must be employed. This has important implications for the dynamics of the network. Second, the power consumption of each vehicle plays a pivotal role in mission run time. An effective communication strategy must take this into account. The final characteristic, unlike most broadcast networks, is that vehicles act in a coordinated and cooperative manner. Current research using flow models to direct network traffic considers users competing for network resources [6]. AUV networks differ in the respect that the vehicles will cooperate to optimize system resources. These characteristics are best highlighted when contrasted with pure ALOHA networks where bandwidth efficiency is very low, transmission power does not factor into the model, and users act independently and competitively.

This paper presents a procedure for constructing controllers that limit and route information flow in AUV networks. It uses a dynamic flow model from [2] tailored to specifically address the unique challenges of the acoustic medium. An optimal control problem is formulated on both a finite and infinite time horizon. Unique, piecewise continuous optimal controllers that limit and route traffic are constructed for each case. A receding horizon approach is proposed in the infinite time case [12], [13]. Finally, a stability result for the receding horizon system is presented.

A. Bandwidth Efficiency

The underwater environment is a severely bandwidth restricted environment. In order to network a large number of autonomous underwater vehicles together, a bandwidth efficient strategy is necessary. In the case of a network of vehicles moving in a fixed or slowly changing formation, the structure of the network can be exploited for bandwidth efficiency. Frequency reuse can be employed in these networks, much as it is in cellular networks.

Each vehicle in the network can be allocated a slice of the total bandwidth via frequency division multiplexing. Vehicles far apart from one another in the network can be allocated the same part of the frequency spectrum. In order to prevent interference, vehicles that use the same part of the spectrum can not both directly communicate with a common vehicle. Therefor the networks considered in this paper consist of underwater vehicles that communicate with a few neighbors, then store and forward packets until they reach their destination.

B. Notational Conventions

This section introduces notation and terminology for describing the networks in this paper. A network consists of a set of N nodes, denoted $\mathcal{N} = \{1, 2, \ldots N\}$, and a set of L links, denoted $\mathcal{L} = \{\,(j,k) \mid j,k \in \mathcal{N}, j \neq k\,\}$. There need not be a link for every pair of nodes. The set of downstream nodes for node j is denoted $O(j) = \{\,k \in \mathcal{N} \mid (j,k) \in \mathcal{L}\,\}$. The set of upstream nodes for node j is denoted $I(j) = \{\,i \in \mathcal{N} \mid (i,j) \in \mathcal{L}\,\}$. Related concepts are the sets of outgoing and incoming links for node j, defined as $\{\,(j,k) \mid k \in O(j)\,\}$ and $\{\,(i,j) \mid i \in I(j)\,\}$, respectfully.

A cycle, $\mathcal{C}$, refers to a set of nodes $\{\,j_c \mid c = 1, 2, \ldots C, (j_c, j_{c+1}) \in \mathcal{L}, j_1 = j_C\,\}$. The integer C is called the size of the cycle and is also denoted by $|\mathcal{C}|$. The notation $\mathcal{C}$ is slightly abused by also denoting the set of links $\{\,(j_c, j_{c+1}) \mid c = 1, 2, \ldots C, (j_c, j_{c+1}) \in \mathcal{L}, j_1 = j_C\,\}$. The set to which $\mathcal{C}$ refers should be obvious from context by examining its elements.

When dealing with networks, the states and parameters in the system are denoted with subscripts whereas symbols without subscripts denote vectors. For example, α_{jk} is the proportion of the traffic arriving at node j that is routed to node k. When the subscript is dropped, it refers to a vector containing the subscripted elements. Therefore α_{jk} are the elements of the vector α.

II. Single Destination Routing

Before considering a network where each AUV can communicate with any other, we first consider single destination routing. In single destination routing, one, and only one, AUV acts as a sink for all of the data entering the network. For example, consider a network of AUVs whose primary task is to gather data and relay it to a centralized location, such as a support ship or buoy, for storage and analysis. Each AUV sends packets of data via acoustic communication channels to neighboring vehicles until the data reaches its final destination.

The AUVs can be modelled as the nodes of a communication network where packets queue while they wait to be transmitted. The model chosen for the queue at each node is based on the following scalar dynamic flow model from [2]

$$\dot{\xi}(\tau) = -\mu G(\xi(\tau)) + \lambda(\tau).$$

The state $\xi(\tau)$ represents the average length of the queue and $\lambda(\tau)$ is the average rate at which traffic enters the queue. The parameter μ is the capacity of the server and the function $G(\xi(\tau))$ is called the utilization function. The utilization function represents the average percentage of capacity being used at each instant.

The utilization $G(\,\cdot\,)$ is modelled as a function of the state. The utilization of a queue switches between 0 and μ depending on whether the queue is empty or not. Intuitively, if the queue contains many elements then its utilization will be equal to μ for a while and one would expect the average utilization to be near μ. If the queue contains few elements, one would expect the average utilization to be near zero. Analytical and empirical methods can be employed to find an appropriate form for $G(\xi)$ [2]. Typically, $G(\xi)$ is a monotonically increasing, concave function of ξ satisfying $G(0) = 0$, $G'(0) = 1$, and $G(\xi) \to 1$ as $\xi \to \infty$.

The current problem of course is not managing a group of individual queues, but systemically controlling a network of queues. At the network level, the interaction dynamics of the queues in the network are primarily influenced by the fact that AUVs communicate in a broadcast environment. Specifically, the network of queues is modelled with the system of equations

$$\begin{aligned} \dot{\xi}_j(\tau) &= -\mu_j G(\xi_j(\tau)) + \phi_j(\tau) r_j(\tau) + \sum_{i \in I(j)} \alpha_{ij}(\tau) \mu_i G(\xi_i(\tau)) \\ \xi_j(t) &= x_j. \end{aligned} \tag{1}$$

The state, ξ_j, represents the length of the queue at node j. The incoming traffic to node j is the sum of two traffic flows; the flow $r_j(\tau)$ is new traffic to the network while $\sum \alpha_{ij}(\tau)\mu_i G(\xi_i(\tau))$ is the sum of traffic arriving from other nodes. Also note that the initial time and state are denoted by t and x_j.

The cooperation between vehicles is used to implement a bandwidth efficient communication strategy. The capacity, μ_j, of each vehicle is determined by the frequency allocated to it. It is assumed that the vehicles not only cooperate in sharing frequency, but also coordinate to maintain a formation. At least, it is assumed that vehicle formation, if not maintained, changes slowly. Vehicles far away from one another can then use same the frequency channels since it is envisioned the vehicles will only communicate with neighbors. Communicating only with neighbors also increases power efficiency since power decreases quadratically with distance.

The control over the network is exercised by routing and limiting the traffic flow. The capacity at each node can be divided amongst outgoing links using time division multiplexing. In this manner, the flow from each node can be proportionally routed onto outgoing links. The routing control α_{jk} is the proportion of traffic flow emanating from node j that is directed to node k. Since all the traffic leaving node j must be directed somewhere, α must satisfy the conservation constraint

$$\sum_{k \in O(j)} \alpha_{jk} = 1 \qquad \forall j \in \mathcal{N}. \tag{2}$$

The control $\phi_j(\tau)$ is the proportion of $r_j(\tau)$, the new traffic entering at node j, that is accepted. Since both α and ϕ are proportions they must satisfy

$$0 \leq \phi_j \leq 1 \qquad \forall j \in \mathcal{N} \tag{3}$$

$$0 \leq \alpha_{jk} \leq 1 \qquad \forall (j,k) \in \mathcal{L}. \tag{4}$$

Finally, we have a set of cycle constraints to eliminate nodes from being disconnected from the rest of the network via cycles.

$$\sum_{(j,k) \in \mathcal{C}} \alpha_{jk} \leq |\mathcal{C}| - 1 \tag{5}$$

This should hold for all possible cycles. In practice heuristics enforce these constraints since there may be too many constraints to enforce explicitly.

We seek a cost criteria that will maximize throughput while minimizing power consumption. Let d_{jk} represent the cost incurred per packet, in terms of power consumption, from using link (j,k) and let b_j represent the relative benefit of accepting traffic at node j. An appropriate cost functional to maximize is

$$J(\alpha,\phi;T) := \int_t^{t+T} \sum_{j\in\mathcal{N}} b_j\phi_j(\tau)r_j(\tau) - \sum_{(j,k)\in\mathcal{L}} \alpha_{jk}(\tau)d_{jk}\mu_j G(\xi_j(\tau))\, d\tau. \tag{6}$$

The first term rewards accepting traffic into the network while the second term penalizes the user for power consumption.

A. Finite Time Optimal Control

Maximizing the cost functional with respect to the dynamic and control constraints defines a finite time optimal control problem. For this problem there exist unique piecewise continuous optimal controllers. These controllers are unique in the sense that other optimal controllers differ only on a set of measure zero. In this section, the piecewise continuous optimal controllers are constructed.

For completeness, we restate the entire finite time optimal control problem defined by the cost criteria, dynamic constraints, and control constraints.

$$\max_{\alpha,\phi\in\Omega} \int_t^{t+T} \sum_{j\in\mathcal{N}} b_j\phi_j(\tau)r_j(\tau) - \sum_{(j,k)\in\mathcal{L}} \alpha_{jk}(\tau)d_{jk}\mu_j G(\xi_j(\tau))\, d\tau \tag{7}$$

subject to

$$\dot{\xi}_j(\tau) = -\mu_j G(\xi_j(\tau)) + \phi_j(\tau)r_j(\tau) + \sum_{i\in I(j)} \alpha_{ij}(\tau)\mu_i G(\xi_i(\tau))$$
$$\xi_j(t) = x_j \tag{8}$$

where Ω is the set of ϕ and α which satisfy

$$\begin{aligned} \sum_{k\in O(j)} \alpha_{jk} &= 1 && \forall j\in\mathcal{N} \\ 0 \le \phi_j &\le 1 && \forall j\in\mathcal{N} \\ 0\le \alpha_{jk} &\le 1 && \forall (j,k)\in\mathcal{L} \\ \sum_{(j,k)\in\mathcal{C}} \alpha_{jk} &\le |\mathcal{C}|-1 && \forall \mathcal{C}. \end{aligned} \tag{9}$$

Traditional optimal control techniques can now be applied. Forming the Hamiltonian

$$H(\xi,\rho,\phi,\alpha) = \sum_{(j,k)\in\mathcal{L}} \alpha_{jk}(d_{jk}+\rho_k)\mu_j G(\xi_j) + \sum_{j\in\mathcal{N}} \phi_j(\rho_j-b_j)r_j - \sum_{j\in\mathcal{N}} \rho_j\mu_j G(\xi_j) \tag{10}$$

and differentiating with respect to ξ yields the costate equations along with its transversality condition

$$\dot{\rho}_j(\tau) = \left(\mu_j \frac{dG}{d\xi_j}(\tau)\right)\left[\rho_j(\tau) - \sum_{k\in O(j)} \alpha_{jk}(\tau)(d_{jk}+\rho_k(\tau))\right]$$
$$\rho_j(t+T) = 0. \tag{11}$$

The Hamiltonian can be split into two linear programming problems

$$\min_{\phi\in\Omega} \sum_{j\in\mathcal{N}} \phi_j(\rho_j-b_j)r_j := \min_{\phi\in\Omega} \sum_{j\in\mathcal{N}} \phi_j H_j \tag{12}$$

$$\min_{\alpha\in\Omega} \sum_{(j,k)\in\mathcal{L}} \alpha_{jk}\mu_j G(\xi_j)(d_{jk}+\rho_k) := \min_{\alpha\in\Omega} \sum_{(j,k)\in\mathcal{L}} \alpha_{jk}H_{jk} \tag{13}$$

whose solutions are naturally binary [3]. This means that if the minimizer of the Hamiltonian is unique, then the value of the optimal controllers will be binary. In the case where the value of the controller is not uniquely determined by the Hamiltonian, the controller may not be binary and another constraint is needed to determine its value. For illustrative purposes, this paper considers the case where only one component of the control will be undetermined at any time. The more general case of multiple undetermined controls is more tedious but conceptually is not any more complex.

Consider the case where a component of ϕ is undetermined. It is not difficult to see that a necessary condition for ϕ^* to minimize (12) is

$$\phi_j^* = \begin{cases} 0 & \text{if } \rho_j - b_j > 0 \\ 1 & \text{if } \rho_j - b_j < 0 \end{cases} \tag{14}$$

In particular, (12) does not have a unique minimizer if $\rho_{\bar{\jmath}} = b_{\bar{\jmath}}$ for some $\bar{\jmath}$. The set

$$\mathcal{S} = \{\,(\tau,\xi,\rho) \mid \gamma(\tau,\xi,\rho) = \rho_{\bar{\jmath}} - b_{\bar{\jmath}} = 0\,\} \tag{15}$$

can be interpreted as a smooth surface in (τ,ξ,ρ) space on which ϕ is not uniquely determined by the Hamiltonian. If a trajectory,$(\tau,\xi(\tau),\rho(\tau))$, hits $\mathcal{S}$ at time $\bar{\tau}$, and is to remain on this surface for some interval $I=[\bar{\tau},\bar{\tau}+\Delta]$, necessary conditions for the trajectory are

$$\frac{d^2}{d\tau^2}\gamma(\tau,\xi(\tau),\rho(\tau)) = \frac{d}{d\tau}\gamma(\tau,\xi(\tau),\rho(\tau)) = 0 \qquad \tau\in I. \tag{16}$$

These additional constraints imply that $\phi_{\bar{\jmath}}$ satisfy the conditions

$$\dot{\rho}_{\bar{\jmath}}(\tau) = \left(\mu_{\bar{\jmath}} \frac{dG}{d\xi_{\bar{\jmath}}}(\tau)\right)\left[\rho_{\bar{\jmath}}(\tau) - \sum_{k\in O(\bar{\jmath})} \alpha_{\bar{\jmath}k}(\tau)(d_{\bar{\jmath}k}+\rho_k(\tau))\right] = 0 \tag{17}$$

$$\ddot{\rho}_{\bar{\jmath}}(\tau) = \phi_{\bar{\jmath}}h_1(\xi,\rho,\alpha) + h_2(\xi,\rho,\alpha) = 0 \tag{18}$$

on I. The functions h_1 and h_2 depend only on the value of known states and controls. Their calculation is straightforward but too lengthy to be practical to report here and can be found in [7]. Equation (18) is linear in $\phi_{\bar{\jmath}}$ and can be solved for $\phi_{\bar{\jmath}}(\tau)$ on I. In fact, $\phi_{\bar{\jmath}}(\tau)$ solved for in this way will be continuous on I.

If either (17) is violated or (18) can not be solved for $\phi_{\bar{\jmath}} \in [0,1]$, then the trajectory will pass through $\mathcal{S}$. According to (14), when the trajectory passes through $\mathcal{S}$, there will be a switching in the control $\phi_{\bar{\jmath}}$. Therefore, $\mathcal{S}$ is referred to as a switching surface.

In this manner, one can construct a piecewise continuous optimal controller, $\phi^*(\tau; t, x)$ for $\tau \in [t, t+T]$, that is uniquely defined given an initial condition (t, x). In order to maintain uniqueness at switching points (where any value of the control can be chosen without losing optimality), continuity from the right can be enforced. In a directly analogous manner, it is possible to construct the unique, piecewise continuous, optimal controller $\alpha^*(\tau; t, x)$ for the finite time problem. Let

$$\mathcal{T} = \{ (\tau, \xi, \rho) \mid \psi(\tau, \xi, \rho) = 0 \} \tag{19}$$

denote the switching surface for α. Additionally, the trajectories given by the controls are continuous in initial conditions (t, x) [4].

B. Receding Horizon Control

The running time and traffic load of an AUV sampling mission is generally not known in advance. This precludes an explicit off-line application of a finite time approach to network control. Solving the infinite time problem, in the absence of steady-state load conditions, also requires knowledge of the load for all time. An alternative solution is to take a receding horizon approach to construct a feedback controller.

Receding horizon controllers are motivated by the following control procedure. At time t, estimate the traffic load over the interval $[t, t+T]$. Solve the finite time problem on this interval and apply the optimal controller until time $t+\delta$, where $\delta < T$. At time $t+\delta$, repeat the process by solving the finite time problem over $[t+\delta, t+\delta+T]$ and apply the optimal controller until time $t+2\delta$. Using this iterative approach, one can control the network over an infinite interval without ever estimating the load more than T seconds into the future.

The controller obtained by letting $\delta \to 0$ is the receding horizon controller. In this section, it is shown that properly defined receding horizon controllers for our system are unique and piecewise continuous. Stabilizing properties of the receding horizon controller are also explored. The chief advantage of this approach is that only knowledge of the load in the near future is required.

It is helpful to introduce vector notation for the finite time system

$$\dot{\xi} = f(\tau, \xi, \phi, \alpha); \qquad \xi(t) = x \tag{20}$$

$$\dot{\rho} = g(\tau, \xi, \rho, \phi, \alpha); \qquad \rho(t+T) = 0. \tag{21}$$

The receding horizon system is defined as

$$\dot{x} = \hat{f}(t, x) = f(t, x, \Phi(t, x), A(t, x)); \qquad x(t_0) = x_0 \tag{22}$$

where $\Phi(t, x)$ and $A(t, x)$ denote the receding horizon controllers. Typically, the receding horizon controllers would be defined as

$$\Phi_j(t, x) = \phi_j^*(t; t, x) \tag{23}$$

$$A_{jk}(t, x) = \alpha_{jk}^*(t; t, x). \tag{24}$$

At every instant t the receding horizon controller optimizes the finite time problem over $[t, t+T]$. However, the receding horizon controller defined in this way does not guarantee existence or uniqueness of solutions. There are surfaces in the state space on which special care must be taken. A suitable definition for the receding horizon controller on these surfaces to guarantee existence and uniqueness is presented.

First, it is useful to consider the cost functional as a function of the initial conditions of the finite time system. Given an initial condition (t, x, p_0) for the finite time system, we can integrate the system to produce the finite time trajectory, $\xi(\tau; t, x)$, and controls, $\phi(\tau; t, x)$ and $\alpha(\tau; t, x)$, over $[t, t+T]$. In turn the trajectory and controls can be integrated by the cost functional to give a scalar cost. In this way, the cost functional can be thought of as a map from the initial conditions to a scalar. We can therefor denote the cost functional as $J(t, x, p_0)$. In fact the cost functional is continuously dependent on the initial conditions [7]. This follows from the fact that the trajectories of the finite time problem are continuously dependent on initial conditions. That is, a pair of initial conditions can be chosen close enough such that their corresponding trajectories are arbitrarily close on the interval. It follows that their costs will also be arbitrarily close since the cost functional is a continuous measure of trajectories.

The following lemma is useful in establishing existence and uniqueness results.

Lemma 1: The function $p(t, x) := \rho^*(t; t, x)$ is continuous in (t, x).

Proof Consider the cost functional as a function of the initial conditions t, x, and p_0. For a fixed t and x the trajectories of the system, and therefore the cost, will vary as the initial costate changes. One could then consider the finite time optimal control problem as that of finding the best initial costate. Consequently,

$$p(t, x) = \rho^*(t; t, x) \in \arg\max_{p_0} J(t, x, p_0). \tag{25}$$

We can conclude that $J(t, x, p(t, x)) - J(t, x, p) > 0$ for all $p \neq p(t, x)$.

Now assume to the contrary that $p(t, x)$ is not continuous. Consider a sequence $(t_k, x_k) \to (t, x)$ such that $p(t_k, x_k) \to p \neq p(t, x)$. Define the difference $d = J(t, x, p(t, x)) - J(t, x, p) > 0$ and construct the real sequence

$$J(t_k, x_k, p(t, x)) - J(t_k, x_k, p(t_k, x_k)) \to d. \tag{26}$$

This implies that there exists an integer K such that for all $k > K$

$$J(t_k, x_k, p(t,x)) - J(t_k, x_k, p(t_k, x_k)) > \frac{d}{2} > 0. \qquad (27)$$

This is a contradiction and the result follows. □

We now use this result to prove a weak local existence and uniqueness result for the receding horizon system.

Lemma 2: Given an initial condition (t_0, x_0) satisfying

$$\hat{\gamma}(t_0, x_0) := \gamma(t_0, x_0, p(t_0, x_0)) \neq 0 \qquad (28)$$

and

$$\hat{\psi}(t_0, x_0) := \psi(t_0, x_0, p(t_0, x_0)) \neq 0 \qquad (29)$$

there exists a $\delta > 0$ such that a unique solution, $x(t)$, of (22) exists on the interval $[t_0, t_0 + \delta]$.

Proof At time t_0, the receding horizon controller $\Phi(t_0, x_0)$ takes on the value of the finite time optimal controller $\phi^*(\tau; t_0, x_0)$ evaluated at $\tau = t_0$. Since (t_0, x_0) satisfies

$$\hat{\gamma}(t_0, x_0) = \gamma(t_0, \xi(t_0; t_0, x(t_0)), \rho(t_0; t_0, x(t_0))) \neq 0 \qquad (30)$$

$\phi^*(t_0; t_0, x_0)$, and therefore $\Phi(t_0, x_0)$, will be binary. Furthermore, (30) will be satisfied in some small neighborhood of (t_0, x_0) since $\hat{\gamma}(t,x)$ is continuous in (t,x). This implies that if $\Phi(t_0, x_0)$ is zero (or one), then $\Phi(t,x)$ will remain zero (or one) in some sufficiently small neighborhood of (t_0, x_0). That is, $\Phi(t,x)$ will be constant in some neighborhood of (t_0, x_0). The same argument for $A(t,x)$, with respect to (29), implies $A(t,x)$ is constant near (t_0, x_0). This implies local existence and uniqueness for trajectories of (22) for any initial condition satisfying (28) and (29). □

In fact, the stronger statement that solutions can be continued up to the surfaces

$$\hat{\gamma}(t,x) = \gamma(t, x, \rho(t,x)) = 0 \qquad (31)$$

$$\hat{\psi}(t,x) = \psi(t, x, \rho(t,x)) = 0 \qquad (32)$$

is true. However, these are still fairly weak existence results since they do not hold for arbitrary initial conditions. The central question is whether a trajectory of the receding horizon system can be continued once it intersects one of these surfaces. The answer is affirmative and trajectories can be continued in much the same manner as they were in the finite time case.

Theorem 1: Define the receding horizon controllers as

$$\Phi(t,x) = \phi^*(t; t, x) \qquad \forall (t,x) \notin \widehat{\mathcal{S}} \qquad (33)$$

$$A(t,x) = \alpha^*(t; t, x) \qquad \forall (t,x) \notin \widehat{\mathcal{T}} \qquad (34)$$

where $\widehat{\mathcal{S}} = \{\, (t,x) \mid \hat{\gamma}(t,x) = 0\}$ and $\widehat{\mathcal{T}} = \{\, (t,x) \mid \hat{\psi}(t,x) = 0\}$. On the sets $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{T}}$ define $\Phi(t,x)$ and $A(t,x)$ implicitly by

$$\frac{d}{dt}\hat{\gamma}(t, x, \Phi, A) = 0 \qquad \forall (t,x) \in \widehat{\mathcal{S}} \quad \Leftrightarrow$$
$$\left(\frac{\partial \gamma}{\partial r}\dot{r} + \frac{\partial \gamma}{\partial p}\frac{\partial p}{\partial t}\right) + \left(\frac{\partial \gamma}{\partial x} + \frac{\partial \gamma}{\partial p}\frac{\partial p}{\partial x}\right) f(t, x, \Phi, A) = 0 \qquad (35)$$

and

$$\frac{d}{dt}\hat{\psi}(t, x, \Phi, A) = 0 \qquad \forall (t,x) \in \widehat{\mathcal{T}} \quad \Leftrightarrow$$
$$\left(\frac{\partial \psi}{\partial r}\dot{r} + \frac{\partial \psi}{\partial p}\frac{\partial p}{\partial t}\right) + \left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial p}\frac{\partial p}{\partial x}\right) f(t, x, \Phi, A) = 0 \qquad (36)$$

respectively. If $p(t,x)$ is C^1, then, given an initial condition (t_0, x_0), there exists a unique solution, $x(t)$, of (22) for all $t \geq t_0$.

Remark 1 We have shown that $p(t,x)$ is continuous, but here it is assumed C^1 so that (35) and (36) may be solved for Φ and A. In the case of linear system, where explicit knowledge of the solution yields a closed form solution of the two point boundary problem, $p(t,x)$ is C^∞. As an illustrative example of how one might solve for the receding horizon controllers, consider (35) in conjunction with (15) and

$$\hat{f}_{\bar{\jmath}}(t,x) = -\mu_{\bar{\jmath}} G(x_{\bar{\jmath}}) + \Phi_{\bar{\jmath}} r_{\bar{\jmath}} + \sum_{i \in I(\bar{\jmath})} \alpha_{i\bar{\jmath}} \mu_i G(\xi_i(\tau)). \qquad (37)$$

The three equations are linear in Φ and when solved yield

$$\Phi_{\bar{\jmath}} = \left(-\mu_{\bar{\jmath}} G(x_{\bar{\jmath}}) + \sum_{i \in I(\bar{\jmath})} \alpha_{i\bar{\jmath}} \mu_i G(\xi_i(\tau))\right) r_{\bar{\jmath}}^{-1}$$
$$- \left(\frac{\partial \gamma}{\partial r_{\bar{\jmath}}}\dot{r}_{\bar{\jmath}} + \frac{\partial \gamma}{\partial p}\frac{\partial p}{\partial t}\right) / \left(\frac{\partial \gamma}{\partial x} + \frac{\partial \gamma}{\partial p}\frac{\partial p}{\partial x} r_{\bar{\jmath}}\right) \qquad (38)$$

Although explicit expressions for the first order derivatives of p are not known, their values can be be calculated numerically.

Remark 2 As in the finite time case, a solution of (35) for Φ (respectively, (36) for A) may not be in the range $[0,1]$. If this is the case, then the trajectory will pass through $\widehat{\mathcal{S}}$ (respectively, $\widehat{\mathcal{T}}$). If the trajectory does remain on $\widehat{\mathcal{S}}$ for some interval, then the solution (38) is continuous on that interval. In general, (35) and (36) give linear equations in the unknown control and can be solved continuously for their values. This implies that, as in the finite time case, one can construct unique, piecewise continuous receding horizon controllers. It is somewhat remarkable that the structure of the receding horizon controller is no more complicated than that of the finite case under the mild assumption that $p(t,x)$ is C^1.

We now turn our attention to a stability result for the receding horizon system. We borrow the following concept of stability from [1].

Definition 1: A solution, $\bar{x}(t)$, of $\dot{x} = \hat{f}(t,x), x(t_0) = \bar{x}_0$ is ultimately bounded if there exist positive constants b and c, and for every $\delta \in (0, c)$ there is a positive constant t_1 such that

$$\|x_0 - \bar{x}_0\| \leq \delta \Rightarrow \|x(t) - \bar{x}(t)\| \leq b, \qquad \forall t \geq t_0 + t_1 \qquad (39)$$

where $x(t)$ is the solution of $\dot{x} = \hat{f}(t,x), x(t_0) = x_0$. The solution $\bar{x}(t)$ is said to be globally uniformly ultimately bounded if this holds for arbitrarily large δ. The solutions of $\dot{x} = \hat{f}(t,x)$ are said to be (globally) uniformly ultimately bounded if every solution of $\dot{x} = \hat{f}(t, x)$ is (globally) uniformly ultimately bounded.

Theorem 2: The solutions to the receding horizon system $\dot{x} = \hat{f}(t, x)$ are globally uniformly ultimately bounded. *Proof* We will show that an arbitrary solution, $\bar{x}(t)$, with initial condition $\bar{x}(t_0) = \bar{x}_0$, is globally uniformly ultimately bounded. Let $x(t)$ be another solution of $\dot{x} = \hat{f}(t, x)$ with $x(t_0) = x_0$. We have for each node j that

$$\begin{aligned}\dot{x}_j(t) &= -\mu_j G(x_j(t)) + \Phi_j(t, x(t)) r_j(t) \\ &\quad + \sum_{i \in I(j)} A_{ij}(t, x(t)) \mu_i G(x_i(t)) \\ &\leq -\mu_j G(x_j(t)) + R + \mu \end{aligned} \tag{40}$$

and

$$\begin{aligned}\dot{\bar{x}}_j(t) &= -\mu_j G(\bar{x}_j(t)) + \Phi_j(t, \bar{x}(t)) r_j(t) \\ &\quad + \sum_{i \in I(j)} A_{ij}(t, \bar{x}(t)) \mu_i G(\bar{x}_i(t)) \\ &\geq -\mu_j G(\bar{x}_j(t)) \end{aligned} \tag{41}$$

where R is a bound on the flow $r(\tau)$ and μ is a bound on the capacities of the network. Bounding the difference using the Lipschitz property (with constant one) of G yields

$$\begin{aligned}|\dot{x}_j(t) - \dot{\bar{x}}_j(t)| &= -\mu_j \left|G(x_j(t)) - G(\bar{x}_j(t))\right| + |R + \mu| \\ &\leq -\mu_j |x_j(t) - \bar{x}_j(t)| + |R + \mu|. \end{aligned} \tag{42}$$

By the comparison lemma,

$$|x_j(t) - \bar{x}_j(t)| \leq |y_j(t) - \bar{y}_j(t)| \tag{43}$$

where

$$\dot{y}_j(t) = -\mu_j y_j(t) + R + \mu \tag{44}$$

$$\dot{\bar{y}}_j(t) = -\mu_j \bar{y}_j(t). \tag{45}$$

We can solve for $\Delta y_j(t) := y_j(t) - \bar{y}_j(t)$ explicitly as

$$\Delta y_j(t) = \left[\Delta y_j(t_0) - \frac{R + \mu}{\mu_j}\right] e^{-\mu_j t} + \frac{R + \mu}{\mu_j}. \tag{46}$$

Letting

$$b = R + \mu + 1 \tag{47}$$

implies that there exists a t_1 such that

$$|x_j(t) - \bar{x}_j(t)| \leq |y_j(t) - \bar{y}_j(t)| \leq b \qquad \forall t > t_0 + t_1. \tag{48}$$

Since this holds for all j, we have

$$\|x(t) - \bar{x}(t)\| \leq b \tag{49}$$

which completes the proof. □

Remark 1 A closely related concept to that of ultimate bounded-ness is input-to-state stability. A corollary of the above theorem is that the receding horizon system is input-to-state stable when $r(t)$ is regarded as the input to the system.

Remark 2 Stability and asymptotic stability of receding horizon controller under certain conditions is reported in [13]. One of the conditions in [14] prohibits the trajectories of the receding horizon system to remain on $\widehat{\mathcal{S}}$ or $\widehat{\mathcal{T}}$. The authors of [14] also impose a terminal constraint on the finite time system which is not practical here. These differences explain the discrepancy between receding horizon stability results reported in the literature and those presented here.

III. General Routing

Thus far, we have considered only the case where one vehicle acts as a sink for all the traffic of the network. Formulating the general routing problem with a standard dynamic flow model is difficult because it requires an expansion of the state space. A state variable is required at each node for each possible destination.

Let ξ_j^k be the number of packets at node j whose final destination is node k. The dynamics for the multi-destination case can be derived by taking the viewpoint of a node in the network. The node receives packets from two sources. The first source is new traffic to the network. The second source is traffic from other nodes in the network. Letting r_j^k be the flow of traffic entering the network at node j and leaving at node k, the dynamics can be written as

$$\dot{\xi}_j^k(\tau) = -\mu_j G(\xi_j) + \phi_j(\tau) r_j^k(\tau) + \sum_{i \in I(j)} \omega_i^k(\tau) \alpha_{ij}(\tau) \mu_i G(\xi_i). \tag{50}$$

The notation ξ_j is used to represent the total number of packets at node j

$$\xi_j = \sum_{l \in \mathcal{N} \setminus j} \xi_j^l. \tag{51}$$

The traffic coming from other nodes is multiplied by a factor ω_i^k. This factor reflects the fact that only a portion of the traffic flowing from node i to j, $\sum \alpha_{ij} \mu_i G(\xi_i)$, has node k as a destination. Therefore ω_i^k is the portion of traffic leaving node i whose destination is node k. A simple yet reasonable model for ω_i^k is the proportion of packets at node i whose destination is node k

$$\omega_i^k = \frac{\xi_i^k}{\xi_i}. \tag{52}$$

This factor adds another nonlinearity to an already nonlinear problem.

Using this model, one could formulate an optimal control problem as in the single destination case. However, this approach has serious computational drawbacks. The most obvious is the increase in the number of state variables. As well as increasing the number of state variables, and just as importantly, this dynamic flow model has increased nonlinearity over its single-destination counterpart. As an alternative, we propose an approximation to the standard dynamic flow model.

A. Forward Dynamic Flow Model

In this subsection a further approximation to the dynamic flow model is introduced to address some of its deficiencies. The key insight is that the time horizon of the input, the average packet flow into the network, is long compared to the time delay experienced by packets from source to destination.

Consider the journey of a packet in the network. It starts at the source, is stored and forwarded by many interim nodes, then exits the network at its destination. The packet is never at more than one node at one time. However, dynamic flow models are not concerned with individual packets, but rather how time averaged packet flow changes. With this in mind, it is possible to send new packets immediately to every node along the path connecting to destination without significantly changing the dynamics of the average queue length. In essence, some node receive new packets a little bit early. The dynamics of the network using this approximation are

$$\dot{\xi}_j(\tau) = -\mu_j G(\xi_j(\tau)) + \phi_j r_j(\tau) + \sum_{(i,k)\in\mathcal{F}_j} r_i^k(\tau) \tag{53}$$

where

$$\mathcal{F}_j = \{\, (i,k) \mid r_i^k \text{ routed through node } j \,\}.$$

Here the parenthetical notation (i,k) represents ordered pairs of nodes, not links. This model of network flow is called the forward dynamic flow model [7] because it immediately forwards each packet to nodes along its path.

At first glance, it may seem that this model does not have any routing controls. In fact, the set $\mathcal{F}_j$ is constructed using the controls. First a constraint on the controls must be mentioned. In order to keep the number of constraints to a minimum, the set of admissible controls is restricted to spanning trees. Restricting the set of admissible controls to spanning trees requires that all links must be bidirectional. This is not a limiting assumption in most broadcast networks and the tradeoff is that cycles are no longer a concern. Spanning trees identify a unique path between any pair of nodes. Given a control vector α representing a spanning tree, the set $\mathcal{F}_j$ can be constructed with an algorithm that finds these unique paths using just matrix multiplications.

The dynamics of the forward dynamic flow model create a problem when formulating the optimal control problem. The fact that only N states are used is fruitful because the goal of the routing and limiting controllers is still to maximize

$$\begin{aligned} J(\alpha,\phi;T) := \int \sum_{j\in\mathcal{N}} b_j\phi_j(\tau) r_j(\tau) \\ - \sum_{(j,k)\in\mathcal{L}} \alpha_{jk}(\tau) d_{jk}\mu_j G(\xi_j(\tau)) - \sum_{j\in\mathcal{N}} \frac{1}{2} a_j \xi_j^2(\tau)\, d\tau \end{aligned} \tag{54}$$

as in the single destination case. The problem is that the Hamiltonian can not be formed using traditional methods. If traditional methods were used, then it would be necessary to minimize

$$\begin{aligned} \bar{H}(\xi,\rho,\phi,\alpha) = \sum_{(j,k)\in\mathcal{L}} \alpha_{jk}(d_{jk}\mu_j G(\xi_j) + \sum_{j\in\mathcal{N}} \frac{1}{2} a_j \xi_j^2 \\ + \sum_{j\in\mathcal{N}} \phi_j(\rho_j - b_j) r_j - \sum_{j\in\mathcal{N}} \rho_j \mu_j G(\xi_j) \\ + \sum_{j\in\mathcal{N}} \rho_j \sum_{(i,k)\in\mathcal{F}_j} r_i^k(\tau). \end{aligned} \tag{55}$$

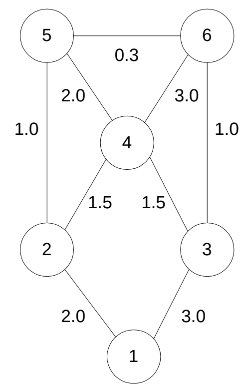

Fig. 1. The six node network for general routing.

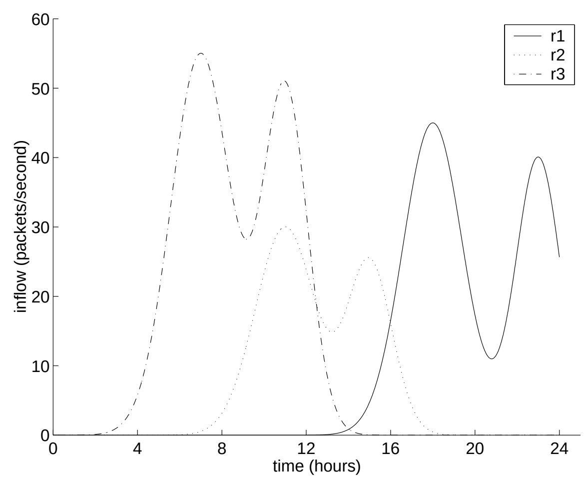

Fig. 2. Load on the receding horizon system.

The final term is implicitly dependent on α which makes the minimization problem very difficult. The good news is that this Hamilton differs from the Hamiltonian of the single destination network only in this term. In fact, the Hamiltonian from the single destination network is the one we use and yields satisfactory results. Finally, by differentiating the Hamiltonian, we use the same costate equations and transversality condition as the single destination case.

B. Simulation Results

In order to verify the above formulation in creating effective routing controllers, the example network of Figure 1 was simulated. The routing controller was implemented in a receding horizon fashion. Every minute, the finite time optimal control problem is solved with a time interval of ten minutes. The optimal controller is used until the next finite time problem is solved.

The load on each node of the network is shown in Figures 2 and 3. The proportion of the flow into each source node

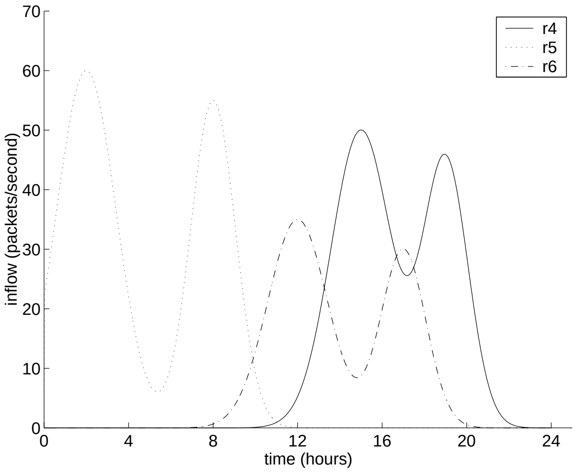

Fig. 3. Load on the receding horizon system.

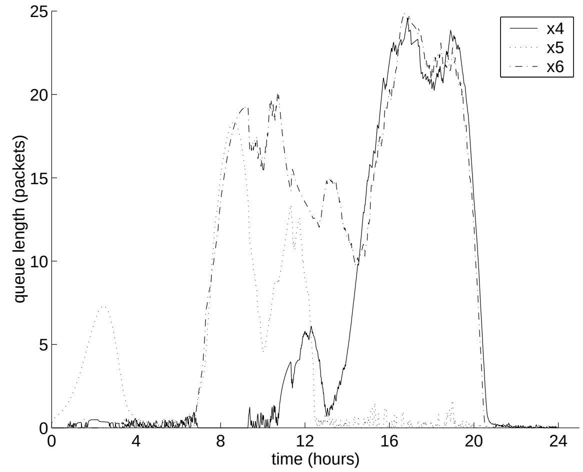

Fig. 5. Queue lengths of the receding horizon system.

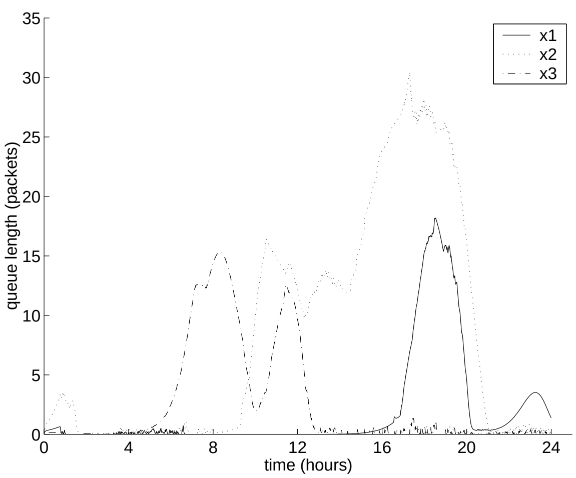

Fig. 4. Queue lengths of the receding horizon system.

that is sent to each destination is given by the matrix:

$$\begin{bmatrix} 0.00 & 0.35 & 0.15 & 0.25 & 0.05 & 0.20 \\ 0.20 & 0.00 & 0.15 & 0.35 & 0.05 & 0.25 \\ 0.35 & 0.25 & 0.00 & 0.20 & 0.05 & 0.15 \\ 0.15 & 0.15 & 0.35 & 0.00 & 0.20 & 0.05 \\ 0.05 & 0.20 & 0.25 & 0.25 & 0.00 & 0.25 \\ 0.20 & 0.20 & 0.20 & 0.05 & 0.35 & 0.00 \end{bmatrix} \tag{56}$$

For example, the proportion of traffic flowing into node 3 that is destined for node 4 is 0.20. The resulting queue lengths are in Figures 4 and 5. The receding horizon controller is able to avoid congestion by rotating the roles of the nodes in the network. Each node is at times an interim node and at times a periphery node. By comparison, under the same load conditions a static spanning tree soon renders the network unstable with queue lengths exceeding 200 packets [7].

IV. Conclusion

The general philosophy behind the modelling and control of the network is to leverage the cooperative structure of an AUV network against the challenges posed by bandwidth and power constraints. Towards that end, this paper presented a model and unique piecewise continuous optimal controllers for directing information flow in AUV networks. Solutions were constructed for both the finite and infinite time horizon. A receding horizon approach was adopted on the infinite horizon and a stability result for it was presented. A chief advantage of receding horizon controllers is that the load need only be estimated into the near future. Also, stability of the receding horizon system is important because receding horizon controllers are necessarily implemented in a discrete, approximate manner.

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