Effect of Buoyancy and Center-of-Mass Offset on Vehicle Drag

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Revision 2
March 8, 2002

The Dorado vehicle typically has a net positive buoyancy, which causes it to achieve a negative pitch angle during steady straight and level flight. This pitch angle, plus the consequent elevator trim cause an increase in drag. The objective of the following analysis is to compute and quantify the increase in drag.

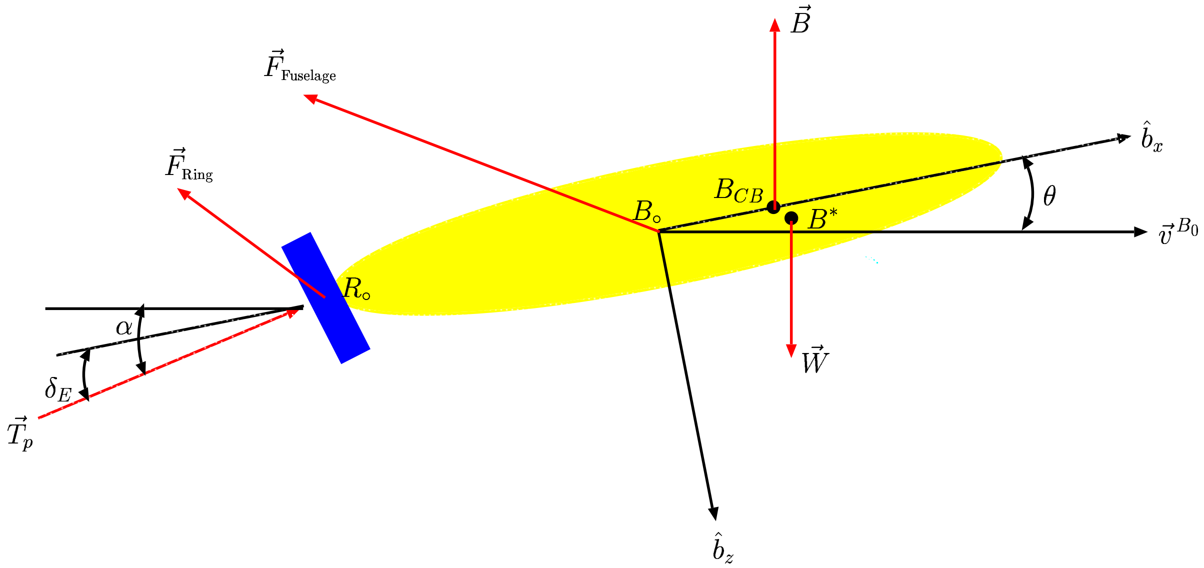

Figure 1: Definition of Reference Frames and Variables

Figure 1 shows the free-body diagram, and defines variables. We will assume the vehicle is in straight and level flight, and is not accelerating in either translation or rotation. The following analysis will examine the planar, longitudinal, force equilibrium equations.

Figure 1 shows the vehicle in a bow-up attitude because this is the direction of positive pitch. For an upward net buoyancy, we would expect the steady-state pitch to be negative.

If U is the forward speed, then

$$\vec{v}^{B_\circ} \triangleq \begin{bmatrix} u \\ v \\ w \end{bmatrix}_B = \begin{bmatrix} U\cos\theta \\ 0 \\ U\sin\theta \end{bmatrix} \tag{1}$$

where $\vec{v}^{\,B_\circ}$ is the velocity of the origin of the body-fixed frame, $B_\circ$, with respect to the inertial frame N, coordinatized in the body-fixed frame, B.

Orthonormal vectors forming the basis of a frame are denoted by hats. For example, the basis of the B frame is $(\hat{b}_x, \hat{b}_y, \hat{b}_z)$. The Newtonian N frame is denoted similarly. Here, the N frame is locally level, that is, N is coincident with B when $\theta = 0$.

For this case, the origin of the body-fixed frame is neither at the center-of-mass nor the center-of-buoyancy. Rather, it is at the vehicle midpoint, $L/2$, which is the origin for the stability derivatives.

The ultimate objective here is to sum forces and torques. First, we must define the various forces and their binding points.

The thrust is:

$$\vec{T}_p \quad = \quad \begin{bmatrix} T_p \cos\delta_E \\ 0 \\ -T_p \sin\delta_E \end{bmatrix}_B \tag{2}$$

where the subscript B indicates coordinatization in the body frame, and

$$|\vec{T}_p| \triangleq T_p \tag{3}$$

We will assume that the thrust, and the lift and drag from the ring, all act at the geometric center of the ring, $R_\circ$. The distance from $B_\circ$ to $R_\circ$ is:

$$\vec{r}^{\,B_\circ R_\circ} = \begin{bmatrix} x_R \\ 0 \\ 0 \end{bmatrix}_B \tag{4}$$

where we take $x_R < 0$.

The force generated by the ring-wing, coordinatized in the B frame, is

$$\vec{F}_{\text{Ring}} = \begin{bmatrix} F_x \\ 0 \\ F_z \end{bmatrix}_B \tag{5}$$

The locations of the center-of-mass, B^*, and the center-of-buoyancy, B_{CB}, in B, are given by:

$$\vec{r}^{\,B_\circ B^*} = \begin{bmatrix} x_G \\ 0 \\ z_G \end{bmatrix}_B \tag{6}$$

$$\vec{r}^{\,B_\circ B_{CB}} = \begin{bmatrix} x_B \\ 0 \\ 0 \end{bmatrix}_B \tag{7}$$

The x axis of the B frame lies along the geometric center-line of the vehicle, and since the center-of-buoyancy is at the volumetric center, the y_B and z_B components are always zero.

The buoyancy is:

$$\begin{aligned} \vec{B} &= -B\hat{n}_z \\ &= \begin{bmatrix} 0 \\ 0 \\ -B \end{bmatrix}_N \end{aligned} \tag{8}$$

Beware that we are using the symbol B to represent both buoyancy (Newtons), and the Body Reference Frame. The meaning is clear from the context.

The weight is:

$$\vec{W} = \begin{bmatrix} 0 \\ 0 \\ W \end{bmatrix}_N \tag{9}$$

Here, W is the weight of the vehicle *including* the weight of the entrained water.

We will assume that the buoyancy B is greater than the weight W. Then,

$$B - W \stackrel{\triangle}{=} \Delta B > 0 \tag{10}$$

The hydrodynamic force acting on the fuselage at $B_\circ$ is given by

$$\vec{F}_{\text{Fuselage}} = \begin{bmatrix} X_{uu}u^2 + X_{ww}w^2 \\ 0 \\ Z_w uw + Z_{w|w|}w|w| \end{bmatrix}_B \tag{11}$$

In general there are many other components of the hydrodynamic force. The other components drop out as a consequence of our assumption of straight and level flight in a vertical plane. See [2] and [1] for a complete analysis of the forces acting on a submarine.

The summation of all force components acting along the body $\hat{b}_z$, or normal, direction, is given by:

$$Z_w uw + Z_{w|w|}w|w| + (W - B)\cos\theta - T_p \sin\delta_E + F_z = 0 \tag{12}$$

The first term is lift, while the second is drag, in the normal direction. The last term is the normal force generated by the ring-wing. Keep in mind that the positive direction is down.

Substituting Equations 1 and 10 into the above,

$$Z_w U^2 \cos\theta \sin\theta + Z_{w|w|}U^2 \sin^2\theta - \Delta B \cos\theta - T_p \sin\delta_E + F_z = 0 \tag{13}$$

We can write a similar equation for the axial direction $\hat{b}_x$:

$$\begin{aligned} 0 &= X_{uu}u^2 + X_{ww}w^2 - (W-B)\sin\theta + T_p\cos\delta_E + F_x \\ &= X_{uu}U^2\cos^2\theta + X_{ww}U^2\sin^2\theta + \Delta B\sin\theta + T_p\cos\delta_E + F_x \end{aligned} \tag{14}$$

The X_{uu} term is viscous drag, and is related to the efficient of drag:

$$X_{uu} = -\frac{1}{2}\rho A C_{Do} \tag{15}$$

where ρ is fluid density, and A is the cross-sectional area.

We can also sum all torques acting about $B_\circ$:

$$\vec{0} = \vec{T}_{\text{Fuselage}} + \vec{r}^{\,B_\circ R_\circ} \times (\vec{F}_{\text{Ring}} + \vec{T}_p) + \vec{r}^{\,B_\circ B^*} \times \vec{W} + \vec{r}^{\,B_\circ B_{CB}} \times \vec{B} \tag{16}$$

where $\vec{T}_{\text{Fuselage}}$ is the hydrodynamic torque acting on the fuselage about $B_\circ$. From [2], the steady-state pitch component is:

$$\vec{T}_{\text{Fuselage}} \cdot \hat{b}_y = M_w uw + M_{w|w|}w|w| \tag{17}$$

Substituting Equations 2 - 9 into 16, coordinatizing in the body frame, and summing the y-axis component with the above yields:

$$\begin{aligned} 0 &= M_w uw + M_{w|w|}w|w| - (x_G W - x_B B)\cos\theta - (z_G W - z_B B)\sin\theta \\ &\quad - x_R(F_z - T_p\sin\delta_E) \\ &= M_w U^2\cos\theta\sin\theta + M_{w|w|}U^2\sin^2\theta - (x_G W - x_B B)\cos\theta \\ &\quad - z_G W\sin\theta - x_R(F_z - T_p\sin\delta_E) \end{aligned} \tag{18}$$

At this point, we have the three nonlinear equations 18, 14, and 13, which may be solved for the three unknowns, (θ, δ_E, T_P). Alternatively, we could solve for U in terms of the thrust T_p, but U has been measured from sea-trial data, and so we'll consider it a known value.

Once (θ, δ_E, T_p) are known, we can easily find the drag in the horizontal plane. But, it is actually T_p that we are interested in, because this translates directly into power:

$$P = T_p U = \eta P_{in} \tag{19}$$

P_{in} is the power supplied by the battery at the thruster motor input. Solving for the three unknowns above will allow us to compute P_{in}, which is equivalent to the range of the vehicle.

η is the propulsion efficiency, and is a difficult number to know, as it includes propeller efficiency, energy lost to heating of the gears and gear box oil, electronic losses in the motor driver electronics, and any other losses that occur in producing thrust. η is best obtained through experiment, which requires mounting the thruster on a test stand instrumented with a load cell, and performing thrust-versus-power measurements in a test tank.

Resuming the solution of the three unknowns, we can find the pitch angle θ by multiplying Eqn. 13 by x_R and adding it to Eqn. 18:

$$\begin{aligned} 0 \;=\; & x_R Z_w U^2 \cos\theta \sin\theta + x_R Z_{w|w|} U^2 \sin^2\theta - x_R \Delta B \cos\theta \\ & + M_w U^2 \cos\theta \sin\theta + M_{w|w|} U^2 \sin^2\theta - (x_G W - x_B B)\cos\theta \\ & - z_G W \sin\theta \end{aligned} \tag{20}$$

The normal force from the ring has fortunately dropped out. Collecting terms above,

$$\begin{aligned} & (x_R Z_w + M_w) U^2 \cos\theta \sin\theta + (x_R Z_{w|w|} + M_{w|w|}) U^2 \sin^2\theta \\ & \quad - (x_R \Delta B + x_G W - x_B B)\cos\theta - z_G W \sin\theta \;=\; 0 \end{aligned} \tag{21}$$

It is interesting that this relation is independent of the other two variables. θ apparently does not depend on the ring deflection δ_E, or any properties of the control surface.

We can reduce the above to an approximate second order equation by replacing the trigonometric functions with their Taylor series equivalents, and retaining only second order terms:

$$\begin{aligned} & (x_R Z_w + M_w) U^2 \theta + (x_R Z_{w|w|} + M_{w|w|}) U^2 \theta^2 \\ & \quad - (x_R \Delta B + x_G W - x_B B)(1 - \frac{\theta}{2}) - z_G W \theta \;=\; 0 \end{aligned} \tag{22}$$

Collecting terms;

$$\theta^2 + \frac{(x_R Z_w + M_w)U^2 - z_G W}{(x_R Z_{w|w|} + M_{w|w|})U^2 + k/2}\theta - \frac{k}{(x_R Z_{w|w|} + M_{w|w|})U^2 + k/2} \;=\; 0 \tag{23}$$

where

$$k \triangleq x_R \Delta B + x_G W - x_B B \tag{24}$$

Solution of the remaining two unknowns requires getting F_x and F_z in terms of (θ, δ_E, T_p). The following several pages are devoted to this.

Let α be the angle of attack on the ring;

$$\alpha = \theta + \delta_E \tag{25}$$

Intuition indicates that θ should be less than zero, while α should be greater than zero. See Figure 1.

For the purposes of analysis, we will assume that the ring lift and drag vectors act at the ring center, which lies along the vehicle center-line. Neglecting transient flow effects, the lift and drag of the ring are given by:

$$L_R = \frac{1}{2}\rho A U^2 C_{LR}(\alpha) \tag{26}$$

$$D_R = \frac{1}{2}\rho A U^2 C_{DR}(\alpha) \tag{27}$$

where

$$C_{LR}(\alpha) = C_{L\alpha R}\alpha + C_{DcR}\alpha|\alpha| \tag{28}$$

$$C_{DR}(\alpha) = C_{DoR} + \frac{C_{LR}(\alpha)^2}{\pi A_{RR} e} \tag{29}$$

See [3], Volume III, pages 306 and 293 for definitions of the nomenclature above. Here, A_{RR} is the aspect ratio, defined as diameter / chord. A_R is the area of the ring, defined as diameter $\times$ chord.

We can find the contribution to lift and drag from the struts in the same manner. The strut lift and drag models are analogous to Equations 26-29, with the subscript R replaced by S, with the exception that A_S is defined as equivalent strut area.

$$A_S \triangleq 2(1 + 2\sqrt{2})A_{SP} \tag{30}$$

where A_{SP} is the planform area of a single strut. The coefficient above exists because there are two struts in the horizontal plane, and four struts at a 45° angle to the horizontal. We assume their components of lift and drag can be obtained by projecting their area into the horizontal plane. Effects of the two vertical struts are neglected.

The total lift and drag are then

$$L = \frac{1}{2}\rho U^2 (A_R C_{LR}(\alpha) + A_S C_{LS}(\alpha)) \tag{31}$$

$$D = \frac{1}{2}\rho U^2 (A_R C_{DR}(\alpha) + A_S C_{DS}(\alpha)) \tag{32}$$

The force applied to the body by the ring, coordinatized in the Newtonian frame, N, is

$$\vec{F}_{\text{Ring}} = \begin{bmatrix} -D \\ L \\ 0 \end{bmatrix}_N \tag{33}$$

Substituting Equations 31 and 32 into 33, and recoordinatizing in the body frame gives

$$\vec{F}_B = \frac{1}{2}\rho U^2 \begin{bmatrix} -\cos\theta(A_R C_{DR}(\alpha) + A_S C_{DS}(\alpha)) + \sin\theta(A_R C_{LR}(\alpha) + A_S C_{LS}(\alpha)) \\ 0 \\ -\sin\theta(A_R C_{DR}(\alpha) + A_S C_{DS}(\alpha)) - \cos\theta(A_R C_{LR}(\alpha) + A_S C_{LS}(\alpha)) \end{bmatrix} \tag{34}$$

We have dropped the "Ring" subscript above to avoid clutter. That is,

$$\left(\vec{F}_{\text{Ring}}\right)_B \triangleq \vec{F}_B \tag{35}$$

Substituting Equations 28-29 and the analogous strut equations into the above gives

$$\vec{F}_B = \frac{1}{2}\rho U^2 \begin{bmatrix} -k_1 \cos\theta\alpha^2 + k_2 \sin\theta\alpha - k_3\cos\theta \\ 0 \\ -k_1 \sin\theta\alpha^2 - k_2\cos\theta\alpha - k_3\sin\theta \end{bmatrix} \tag{36}$$

where

$$k1 \triangleq \frac{A_R C_{L\alpha R}^2}{\pi A_{RR} e} + \frac{A_S C_{L\alpha S}^2}{\pi A_{SR} e} \tag{37}$$

$$k2 \triangleq A_R C_{L\alpha R} + A_S C_{L\alpha S} \tag{38}$$

$$k3 \triangleq A_R C_{DoR} + A_S C_{DoS} \tag{39}$$

We have assumed that the contribution from cross-flow drag is negligible, and so have set

$$C_{DcR} = C_{DcS} = 0 \tag{40}$$

in Equation 28, and the analogous strut equation.

Substituting Equations 36 and 25 into the Axial Force Equation 14 gives

$$\begin{aligned} X_{uu}U^2\cos^2\theta + X_{ww}U^2\sin^2\theta + \Delta B\sin\theta + T_p\cos\delta_E & \\ +\frac{1}{2}\rho U^2(-k_1\cos\theta(\theta+\delta_E)^2 + k_2\sin\theta(\theta+\delta_E) - k_3\cos\theta) &= 0 \end{aligned} \tag{41}$$

Similarly, substituting Equations 36 and 25 into the Normal Force Equation 13 gives

$$\begin{aligned} Z_w U^2\cos\theta\sin\theta + Z_{w|w|}U^2\sin^2\theta - \Delta B\cos\theta - T_p\sin\delta_E & \\ +\frac{1}{2}\rho U^2(-k_1\sin\theta(\theta+\delta_E)^2 - k_2\cos\theta(\theta+\delta_E) - k_3\sin\theta) &= 0 \end{aligned} \tag{42}$$

CONCLUSIONS

The three equations 21, 41, and 42 are nonlinear, but may be numerically solved for (θ, δ_E, T_p). Section A of the appendix furnishes values for all of the parameters involved. Sections B.1 - B.6 contain solutions for the three unknowns, for various buoyancies and vehicle lengths. Buoyancies of 30 and 66 Newtons are measured values. $x_G = x_B$ for all cases in this section.

The numerical solution for δ_E is interesting. We expect that $\Delta B > 0$ implies:

$$\theta < 0 \tag{43}$$
$$\alpha = \delta_E + \theta > 0 \tag{44}$$

That is, it would seem that the ring should have a positive angle of attack α, so that its resultant force is upward, counteracting the vehicle's static restoring moment. It turns out that the lift from the fuselage creates a much larger moment about $B_\circ$ than the restoring force, and it is negative. Thus, the total torque that the ring must counteract is negative, requiring the ring to generate a positive moment, which in turn requires the ring to have a negative angle of attack. In this case, intuition turns out to be wrong, in that Equation 44 is incorrect.

The appendix gives values for the relative sizes of the individual terms of Equations 21, 41, and 42, which justifies the above reasoning.

In the case of $B = 30$ Newtons, the increase in required thrust is only 1.7%, which is not large. However, for $B = 90$ Newtons, the thrust increases by 16.1%, which is significant. Values of B in the range of 60-90 Newtons are probable under the current system of ballasting.

Section C furnishes values for an axial one centimeter center-of-mass offset, in both directions, with the vehicle held neutrally buoyant. That is, $x_G = x_B \pm 1$cm, with $\Delta B = 0$.

In this case, the pitch angle has an interesting and possibly counter-intuitive value. When the vehicle is bow-heavy, it ends up in a bow-up attitude. This is because it is using the lift off the fuselage and the ring to counteract the static negative pitch torque. See Sections C.1 - C.4 for the various combinations of vehicle length and x_G offset.

The increase in thrust is small; $< 2\%$ in all cases.

References

[1] J. Feldman. Dtnsrdc revised standard equations of motion for submarine simulation. Report SPD-0393-09, David W. Taylor Naval Ship Research and Development Center, 1979.

[2] M. Gertler and G. Hagen. Standard equations of motion for submarine simulation. Report 2510, David W. Taylor Naval Ship Research and Development Center, 1967.

[3] Edward V. Lewis, editor. *Principles of Naval Architecture, Second Revision.* Society of Naval Architects and Marine Engineers, third edition, 1989.

Appendix

A Model Values

A.1 Vehicle with Single Mid-body

```
DORADO389 PARAMETERS
-- Geometric Properties --
3.945075       : (len) Length, Meters.
12.943159      : (len) Length, feet.
0.533400       : (dia) Diameter, Meters.
21.000000      : (dia) Diameter, Inches.
-1.972538      : (xR) Distance from Bo to CP of ring, Meters.
0.118109       : (xG) x CG offset, Meters.
0.000000       : (yG) y CG offset, Meters.
0.006800       : (zG) z CG offset, Meters.
0.118109       : (xB) x CB offset, Meters.
0.000000       : (yB) y CB offset, Meters.
0.000000       : (zB) z CB offset, Meters.
-- Control Surface Hydrodynamic Properties --
   - Fins or Struts -
0.010161       : (StrutArea) Control surface Area, Meters^2.
15.750000      : (StrutArea) Control surface Area, Inches^2.
0.049063       : (LinCtrlArea) Equivalent control surface area, Meters^2.
76.047727      : (LinCtrlArea) Equivalent control surface area, Inches^2.
6.280000       : (dCl) Coeff. of lift slope.
0.810000       : (Cdc) Crossflow drag coefficient
0.012000       : (Cd0) Constant component of drag coeff.
3.111111       : (Ar) Ring/strut aspect ratio.
0.900000       : (Ef) Oswald efficiency factor.
   - Ring -
0.048387       : (RingArea) Control surface Area, Meters^2.
75.000000      : (RingArea) Control surface Area, Inches^2.
4.800000       : (dClR) Coeff. of lift slope.
0.810000       : (CdcR) Crossflow drag coefficient
0.010000       : (Cd0R) Constant component of drag coeff.
4.260000       : (ArR) Ring/strut aspect ratio.
```

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0.900000       : (EfR) Oswald efficiency factor.
-- Fuselage Hydrodynamic Properties --
-19.980179     : (Xuu)   2nd. Deriv of X wrto u, kg/m.
-54.270714     : (Xww)   Deriv of X wrto ww, kg/m.
-112.392498    : (Zw)    Deriv of Z wrto w, kg/m.
-598.999761    : (Zw|w|) 2nd. Deriv of Z wrto w, kg/m.
564.420379     : (Mw)    Deriv of M wrto w, kg.
-310.431208    : (Mw|w|) 2nd Deriv of M wrto w, kg.
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A.2 Vehicle with Two Mid-bodies

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DORADO556 PARAMETERS
-- Geometric Properties --
5.554800       : (len) Length, Meters.
18.224409      : (len) Length, feet.
0.533400       : (dia) Diameter, Meters.
21.000000      : (dia) Diameter, Inches.
-2.777400      : (xR) Distance from Bo to CP of ring, Meters.
0.120877       : (xG) x CG offset, Meters.
0.000000       : (yG) y CG offset, Meters.
0.006500       : (zG) z CG offset, Meters.
0.120877       : (xB) x CB offset, Meters.
0.000000       : (yB) y CB offset, Meters.
0.000000       : (zB) z CB offset, Meters.
-- Control Surface Hydrodynamic Properties --
   - Fins or Struts -
0.010161       : (StrutArea) Control surface Area, Meters^2.
15.750000      : (StrutArea) Control surface Area, Inches^2.
0.049063       : (LinCtrlArea) Equivalent control surface area, Meters^2.
76.047727      : (LinCtrlArea) Equivalent control surface area, Inches^2.
6.280000       : (dCl) Coeff. of lift slope.
0.810000       : (Cdc) Crossflow drag coefficient
0.012000       : (Cd0) Constant component of drag coeff.
3.111111       : (Ar) Ring/strut aspect ratio.
0.900000       : (Ef) Oswald efficiency factor.
   - Ring -
0.048387       : (RingArea) Control surface Area, Meters^2.
75.000000      : (RingArea) Control surface Area, Inches^2.
4.800000       : (dClR) Coeff. of lift slope.
0.810000       : (CdcR) Crossflow drag coefficient
0.010000       : (Cd0R) Constant component of drag coeff.
4.260000       : (ArR) Ring/strut aspect ratio.
0.900000       : (EfR) Oswald efficiency factor.
-- Fuselage Hydrodynamic Properties --
-25.502857     : (Xuu)   2nd. Deriv of X wrto u, kg/m.
-67.911429     : (Xww)   Deriv of X wrto ww, kg/m.
-138.000394    : (Zw)    Deriv of Z wrto w, kg/m.
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-920.141796    : (Zw|w|) 2nd. Deriv of Z wrto w, kg/m.
885.188138     : (Mw)    Deriv of M wrto w, kg.
-486.853476    : (Mw|w|) 2nd Deriv of M wrto w, kg.
```

B Thrust Versus Buoyancy

B.1 Single Mid-body, $B = 30$ Newtons

```
 Parameters:
deltaB = 30 Newtons.
Speed  = 1.5433 Meters/second.
xB     = 0.11811 Meters.
xG     = 0.11811 Meters.
zG     = 0.0068 Meters.

Numeric Solution:
Theta  = -2.054 Degrees.
deltaE = 0.12407 Degrees.
Thrust = 49.7182 Newtons.
Nominal Thrust   = 48.9035 Newtons.
Thrust Increase  = 1.6658 %.
Pitch Equation:
   Mlift  = -48.8217 %.
   Mdrag  = -0.96303 %.
   MxDb   = 3.5901 %.
   MzDb   = 1.7471 %.
   MTp    = -0.21532 %.
   MRing  = 44.6627 %.
Axial Force Equation:
   Xlift  = -0.167 %.
   Xdrag  = -47.7988 %.
   XDb    = -1.0813 %.
   XTp    = 50 %.
   XRing  = -0.95288 %.
Normal Force Equation:
   Zlift  = 15.0194 %.
   Zdrag  = -2.8708 %.
   ZDb    = -46.9606 %.
   ZTp    = -0.16864 %.
   ZRing  = 34.9805 %.
```

B.2 Single Mid-body, $B = 66$ Newtons

```
 Parameters:
deltaB = 66 Newtons.
Speed  = 1.5433 Meters/second.
xB     = 0.11811 Meters.
xG     = 0.11811 Meters.
zG     = 0.0068 Meters.

Numeric Solution:
 Theta  = -4.7968 Degrees.
 deltaE = 0.16679 Degrees.
 Thrust = 53.0086 Newtons.
 Nominal Thrust   = 48.9035 Newtons.
 Thrust Increase  = 8.3943 %.
 Pitch Equation:
    Mlift  = -47.6703 %.
    Mdrag  = -2.2002 %.
    MxDb   = 3.3055 %.
    MzDb   = 1.7108 %.
    MTp    = -0.12953 %.
    MRing  = 44.9837 %.
 Axial Force Equation:
    Xlift  = -0.84194 %.
    Xdrag  = -44.0174 %.
    XDb    = -5.1407 %.
    XTp    = 49.374 %.
    XRing  = 0.62602 %.
 Normal Force Equation:
    Zlift  = 14.6955 %.
    Zdrag  = -6.5723 %.
    ZDb    = -43.326 %.
    ZTp    = -0.10166 %.
    ZRing  = 35.3045 %.
```

B.3 Single Mid-body, $B = 90$ Newtons

```
 Parameters:
deltaB = 90 Newtons.
Speed  = 1.5433 Meters/second.
xB     = 0.11811 Meters.
xG     = 0.11811 Meters.
zG     = 0.0068 Meters.

Numeric Solution:
 Theta  = -6.8647 Degrees.
 deltaE = 0.12173 Degrees.
 Thrust = 56.7738 Newtons.
 Nominal Thrust   = 48.9035 Newtons.
 Thrust Increase  = 16.0935 %.
 Pitch Equation:
    Mlift  = -46.8294 %.
    Mdrag  = -3.1007 %.
    MxDb   = 3.0979 %.
    MzDb   = 1.6869 %.
    MTp    = -0.069842 %.
    MRing  = 45.2153 %.
 Axial Force Equation:
    Xlift  = -1.5515 %.
    Xdrag  = -39.411 %.
    XDb    = -9.0375 %.
    XTp    = 47.6974 %.
    XRing  = 2.3026 %.
 Normal Force Equation:
    Zlift  = 14.4586 %.
    Zdrag  = -9.2768 %.
    ZDb    = -40.6683 %.
    ZTp    = -0.054899 %.
    ZRing  = 35.5414 %.
```

B.4 Double Mid-body, $B = 30$ Newtons

```
 Parameters:
deltaB = 30 Newtons.
Speed  = 1.5433 Meters/second.
xB     = 0.12088 Meters.
xG     = 0.12088 Meters.
zG     = 0.0065 Meters.

Numeric Solution:
 Theta  = -1.7803 Degrees.
 deltaE = -0.10235 Degrees.
 Thrust = 62.8245 Newtons.
 Nominal Thrust   = 62.0579 Newtons.
 Thrust Increase  = 1.2353 %.
 Pitch Equation:
    Mlift  = -49.1596 %.
    Mdrag  = -0.8404 %.
    MxDb   = 2.7215 %.
    MzDb   = 1.6204 %.
    MTp    = 0.23405 %.
    MRing  = 45.424 %.
 Axial Force Equation:
    Xlift  = -0.12426 %.
    Xdrag  = -48.2982 %.
    XDb    = -0.74177 %.
    XTp    = 50 %.
    XRing  = -0.83576 %.
 Normal Force Equation:
    Zlift  = 15.8983 %.
    Zdrag  = -3.2949 %.
    ZDb    = -46.7051 %.
    ZTp    = 0.17481 %.
    ZRing  = 33.9269 %.
```

B.5 Double Mid-body, $B = 66$ Newtons

```
 Parameters:
deltaB = 66 Newtons.
Speed  = 1.5433 Meters/second.
xB     = 0.12088 Meters.
xG     = 0.12088 Meters.
zG     = 0.0065 Meters.

Numeric Solution:
 Theta  = -4.2417 Degrees.
 deltaE = -0.35069 Degrees.
 Thrust = 66.0191 Newtons.
 Nominal Thrust   = 62.0579 Newtons.
 Thrust Increase  = 6.3831 %.
 Pitch Equation:
    Mlift  = -48.0404 %.
    Mdrag  = -1.9596 %.
    MxDb   = 2.4577 %.
    MzDb   = 1.5871 %.
    MTp    = 0.34668 %.
    MRing  = 45.6086 %.
 Axial Force Equation:
    Xlift  = -0.66857 %.
    Xdrag  = -45.6432 %.
    XDb    = -3.6882 %.
    XTp    = 49.8784 %.
    XRing  = 0.12166 %.
 Normal Force Equation:
    Zlift  = 15.5799 %.
    Zdrag  = -7.7046 %.
    ZDb    = -42.2954 %.
    ZTp    = 0.25966 %.
    ZRing  = 34.1604 %.
```

B.6 Double Mid-body, $B = 90$ Newtons

```
 Parameters:
deltaB = 90 Newtons.
Speed  = 1.5433 Meters/second.
xB     = 0.12088 Meters.
xG     = 0.12088 Meters.
zG     = 0.0065 Meters.

Numeric Solution:
Theta  = -6.1969 Degrees.
deltaE = -0.62025 Degrees.
Thrust = 69.8346 Newtons.
Nominal Thrust   = 62.0579 Newtons.
Thrust Increase  = 12.5313 %.
Pitch Equation:
   Mlift  = -47.1823 %.
   Mdrag  = -2.8177 %.
   MxDb   = 2.2553 %.
   MzDb   = 1.5636 %.
   MTp    = 0.43783 %.
   MRing  = 45.7432 %.
Axial Force Equation:
   Xlift  = -1.3156 %.
   Xdrag  = -41.9036 %.
   XDb    = -6.7808 %.
   XTp    = 48.7392 %.
   XRing  = 1.2608 %.
Normal Force Equation:
   Zlift  = 15.3351 %.
   Zdrag  = -11.1023 %.
   ZDb    = -38.8977 %.
   ZTp    = 0.32865 %.
   ZRing  = 34.3362 %.
```

C Thrust Versus CG Offset

C.1 Single Mid-body, $x_G = x_B + .01$ Meters

```
 Parameters:
deltaB = 0 Newtons.
Speed  = 1.5433 Meters/second.
xB     = 0.11811 Meters.
xG     = 0.12811 Meters.
zG     = 0.0068 Meters.

Numeric Solution:
 Theta  = 2.1309 Degrees.
 deltaE = -2.9478 Degrees.
 Thrust = 49.5095 Newtons.
 Nominal Thrust   = 48.9035 Newtons.
 Thrust Increase  = 1.2391 %.
 Pitch Equation:
    Mlift  = 33.9979 %.
    Mdrag  = -0.69576 %.
    MxDb   = -48.0876 %.
    MzDb   = -1.2167 %.
    MTp    = 3.4181 %.
    MRing  = 12.584 %.
 Axial Force Equation:
    Xlift  = -0.18073 %.
    Xdrag  = -48.059 %.
    XDb    = 0 %.
    XTp    = 50 %.
    XRing  = -1.7603 %.
 Normal Force Equation:
    Zlift  = -41.7258 %.
    Zdrag  = -8.2744 %.
    ZDb    = 0 %.
    ZTp    = 10.68 %.
    ZRing  = 39.3198 %.
```

C.2 Single Mid-body, $x_G = x_B - .01$ Meters

```
 Parameters:
deltaB = 0 Newtons.
Speed  = 1.5433 Meters/second.
xB     = 0.11811 Meters.
xG     = 0.10811 Meters.
zG     = 0.0068 Meters.

Numeric Solution:
 Theta  = -2.3289 Degrees.
 deltaE = 2.8582 Degrees.
 Thrust = 49.3797 Newtons.
 Nominal Thrust   = 48.9035 Newtons.
 Thrust Increase  = 0.97364 %.
 Pitch Equation:
    Mlift  = -37.5934 %.
    Mdrag  = -0.84091 %.
    MxDb   = 48.6544 %.
    MzDb   = 1.3456 %.
    MTp    = -3.345 %.
    MRing  = -8.2207 %.
 Axial Force Equation:
    Xlift  = -0.21641 %.
    Xdrag  = -48.1685 %.
    XDb    = 0 %.
    XTp    = 50 %.
    XRing  = -1.6151 %.
 Normal Force Equation:
    Zlift  = 50 %.
    Zdrag  = -10.8376 %.
    ZDb    = 0 %.
    ZTp    = -11.3264 %.
    ZRing  = -27.836 %.
```

C.3 Double Mid-body, $x_G = x_B + .01$ Meters

```
 Parameters:
deltaB = 0 Newtons.
Speed  = 1.5433 Meters/second.
xB     = 0.12088 Meters.
xG     = 0.13088 Meters.
zG     = 0.0065 Meters.

Numeric Solution:
 Theta  = 1.9625 Degrees.
 deltaE = -2.8902 Degrees.
 Thrust = 62.7225 Newtons.
 Nominal Thrust   = 62.0579 Newtons.
 Thrust Increase  = 1.0709 %.
 Pitch Equation:
    Mlift  = 32.6387 %.
    Mdrag  = -0.6151 %.
    MxDb   = -48.3089 %.
    MzDb   = -1.0759 %.
    MTp    = 3.973 %.
    MRing  = 13.3883 %.
 Axial Force Equation:
    Xlift  = -0.15141 %.
    Xdrag  = -48.4282 %.
    XDb    = 0 %.
    XTp    = 50 %.
    XRing  = -1.4204 %.
 Normal Force Equation:
    Zlift  = -40.7011 %.
    Zdrag  = -9.2989 %.
    ZDb    = 0 %.
    ZTp    = 11.4422 %.
    ZRing  = 38.5579 %.
```

C.4 Double Mid-body, $x_G = x_B - .01$ Meters

```
 Parameters:
deltaB = 0 Newtons.
Speed  = 1.5433 Meters/second.
xB     = 0.12088 Meters.
xG     = 0.11088 Meters.
zG     = 0.0065 Meters.

Numeric Solution:
 Theta  = -2.2183 Degrees.
 deltaE = 2.7773 Degrees.
 Thrust = 62.568 Newtons.
 Nominal Thrust   = 62.0579 Newtons.
 Thrust Increase  = 0.82203 %.
 Pitch Equation:
    Mlift  = -37.2449 %.
    Mdrag  = -0.79349 %.
    MxDb   = 48.772 %.
    MzDb   = 1.228 %.
    MTp    = -3.8457 %.
    MRing  = -8.116 %.
 Axial Force Equation:
    Xlift  = -0.19389 %.
    Xdrag  = -48.5272 %.
    XDb    = 0 %.
    XTp    = 50 %.
    XRing  = -1.2789 %.
 Normal Force Equation:
    Zlift  = 50 %.
    Zdrag  = -12.9138 %.
    ZDb    = 0 %.
    ZTp    = -11.9233 %.
    ZRing  = -25.1629 %.
```