Probabilistic Analysis of Dvl Errors

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Introduction:

The objective is to compute a upper bound on position error, as a function of distance traveled, as determined from the Dvl. The following analysis takes only Dvl errors into account, and additionally contains several idealizing assumptions.

Conclusions:

The Dvl has a CEP of $\leq .52\%$ of distance traveled, under the following assumptions:

1. The vehicle moves in steady straight and level flight at 1.5 m/s.
2. The scale factor and bias are independent, Gaussian random variables.
3. The scale factor and bias for each Dvl axis are independent.
4. The scale factor and bias values are randomly selected at the outset of each logging session, and remain (approximately) constant during the logging session.
5. The Dvl has a scale factor error with a standard deviation of $\leq .4\%$.
6. The Dvl has a bias of with a standard deviation of $\leq .2$ cm/sec.

At 1.0 meter/second, the Dvl has a CEP of $\leq .58\%$ of distance traveled.

Assumptions 2-6 are implied by the web page "How Accurately Can I Navigate With My Dvl?", at http://www.dvlnav.com/faq/faq.html#six.

Discussion:

First, we will work out the case of a single axis, paralleling the web page. Let v_m be a component of velocity measured by the Dvl, and v be the corresponding actual component.

$$v_m(k) = (1 + X_1)v(k) + X_2 + N(k) \tag{1}$$

X_1 and X_2 are independent zero-mean Gaussian random variables. We will assume that their values are selected at the start of a Dvl logging session, and remain constant thereafter. $(1 + X_1)$ is the scale factor, and X_2 is the bias. The measurement noise $N(k)$ is a sequence of independent zero-mean Gaussian random variables.

The error is then

$$v_e(k) = v_m(k) - v(k) = X_1 v(k) + X_2 + N(k) \tag{2}$$

We will further assume that the Dvl is rigidly mounted in a vehicle which moves in steady straight and level flight during the logging session. Then, the distance traveled is

$$\begin{aligned} D(t_f) &= V t_f & (3) \\ &= VMT_s & (4) \end{aligned}$$

where V is constant, t_f is the final or end time, T_s is the sampling interval, and M is the total number of samples over the duration of the logging session.

Accumulating Eqn. 2 for all M samples,

$$\begin{aligned} D_e(MT_s) &= \sum_{k=1}^{M} v_e(k) T_s & (5) \\ &= \sum_{k=1}^{M} \left(X_1 V + X_2 + N(k)\right) T_s & (6) \\ &= X_1 V T_s M + X_2 T_s M + T_s \sum_{k=1}^{M} N(k) & (7) \end{aligned}$$

Where D_e is the error distance. Substituting 4 gives

$$D_e(MT_s) = X_1 D + X_2 \frac{D}{V} + \frac{D}{VM} \sum_{k=1}^{M} N(k) \tag{8}$$

and finally,

$$\frac{D_e}{D} = X_1 + \frac{X_2}{V} + \frac{1}{VM} \sum_{k=1}^{M} N(k) \tag{9}$$

Since X_1, X_2 and $N(k)$ are all independent and zero-mean,

$$E\{\frac{D_e}{D}\} = E\{X_1\} + \frac{1}{V} E\{X_2\} + \frac{1}{VM} \sum_{k=1}^{M} E\{N(k)\} = 0 \tag{10}$$

It is well known ([1], p. 88) that the sum of independent Gaussian random variables produces a Gaussian random variable whose mean is the sum of the means, and whose variance is the sum of the variances. Further, scaling a Gaussian random variable also produces a Gaussian random variable whose standard deviation is similarly scaled. Thus,

$$E\{\left(\frac{D_e}{D}\right)^2\} \triangleq \left(\frac{\sigma_{D_e}}{D}\right)^2 = E\{(X_1)^2\} + E\{\left(\frac{X_2}{V}\right)^2\} + E\{\left(\frac{1}{VM}\sum_{k=1}^{M} N(k)\right)^2\} \quad (11)$$

$$= \sigma_{X_1}^2 + \left(\frac{\sigma_{X_2}}{V}\right)^2 + \frac{1}{(VM)^2}\sum_{k=1}^{M} E\{N(k)\}^2 \quad (12)$$

$$= \sigma_{X_1}^2 + \left(\frac{\sigma_{X_2}}{V}\right)^2 + \frac{1}{M}\left(\frac{\sigma_N}{V}\right)^2 \quad (13)$$

Substituting the numbers given the Conclusion section, and letting $M \to \infty$,

$$\frac{\sigma_{D_e}}{D} = \sqrt{.004^2 + (.2/150)^2} = .422\% \quad (14)$$

$$\sqrt{.004^2 + (.2/100)^2} = .447\% \quad (15)$$

Thus we can expect the navigation error in a single Dvl axis to be less than .422% of distance traveled for 68% of the trials, at $V = 150$ cm/sec.

One might at first find Eqn. 13 perplexing, because intuition says that there should be a random walk involved, that is, the third term should have the M in the numerator instead of the denominator. In fact, the third term *is* a random walk. Multiplying each side by D^2 and substituting Eqn. 4 yeilds

$$\sigma_{D_e}^2 = (MT_sV\sigma_{X_1})^2 + (MT_s\sigma_{X_2})^2 + M(T_s\sigma_N)^2 \quad (16)$$

The last term now has the expected form of a random walk. Eqn. 16 shows that the other two terms become large faster than the random walk, and so in the limit the contribution from the random walk is insignificant.

Now, we will consider the case of horizontal velocity resolved onto both Dvl x and y axes. Here, suppose that the Dvl is mounted as usual with the alignment notch facing forward. In steady forward flight, velocity is resolved into equal components on each axis,

$$^N\vec{v}^{DVL} = \begin{bmatrix} -V/\sqrt{2} \\ V/\sqrt{2} \end{bmatrix}_{DVL} \quad (17)$$

where $^N\vec{v}^{DVL}$ is the velocity of the (origin of the) Dvl frame with respect to an inertially fixed, or Newtonian, frame N. The DVL subscript on the right hand side indicates coordinatization in that frame.

Under our assumption that the random variables associated with each axis are independent, Eqn. 13 then applies to each axis, but with the component velocity scaled as above:

$$\sigma_{D_{xe}}^2 = \sigma_{X_1}^2 + 2\left(\frac{\sigma_{X_2}}{V}\right)^2 \quad (18)$$

$$\sigma_{D_{ye}}^2 = \sigma_{Y_1}^2 + 2\left(\frac{\sigma_{Y_2}}{V}\right)^2 \quad (19)$$

where the Y_i are analogous random variables for the Dvl y axis.

Since we are really concerned with radial error, it makes sense to define a new random variable that combines the error in each axis into a miss distance,

$$R = \sqrt{D_{xe}^2 + D_{ye}^2} \tag{20}$$

The random variable R has the well-known Rayleigh density function [2], as long as the component Gaussian densities have equal statistics and are independent. This requirement applies here, so let

$$\sigma_{X_1} = \sigma_{Y_1} \tag{21}$$

$$\sigma_{X_2} = \sigma_{Y_2} \tag{22}$$

implying

$$\sigma_{D_{xe}}^2 = \sigma_{D_{ye}}^2 \tag{23}$$

The Rayleigh distribution function (p.195, [2]) is given by

$$P_R(r) = 1 - \exp \frac{-r^2}{\overline{R^2}} \tag{24}$$

where $\overline{R^2}$ is the mean-square value,

$$\overline{R^2} \triangleq E\{R^2\} \tag{25}$$

Beware that $\overline{R^2}$ is *not* the variance, because the Rayleigh density has a non-zero mean. Also,

$$\overline{R^2} = 2\sigma_{D_{xe}}^2 \tag{26}$$

It is common to set the left-hand side of 24 to a value P_o, such as 50% or 95%, and solve for the corresponding radius r_o,

$$r_o = \sqrt{\ln\left(\frac{1}{1-P_o}\right)\overline{R^2}} \tag{27}$$

The Circular Error Probable (CEP) radius, for a given probability P_o, is found by substituting Eqn. 18 into Eqn. 26, and the result into the above:

$$\frac{r_o}{D} = \sqrt{\ln\left(\frac{1}{1-P_o}\right) 2\left(\sigma_{X_1}^2 + 2\left(\frac{\sigma_{X_2}}{V}\right)^2\right)} \tag{28}$$

This means that for many trials of distance D, P_o of the results will fall within radius r_o.

The Circle of Equal Probability (CEP - same as 50% Circular Error Probable) is found by setting $P_o = .5$, and substituting in the numerical values specified in the Conclusion:

$$\begin{aligned} \frac{r_o}{D} &= .52\% \text{ CEP; } V = 150\text{cm/s} && (29) \\ &= .58\% \text{ CEP; } V = 100\text{cm/s} && (30) \end{aligned}$$

Thus for many 1km missions, about 50% of them will have less than 5.2 meters radial error, or .52% of distance traveled.

We can also select $P_o = 68.27\%$, so that we can compare the result to the single-axis case;

$$\begin{aligned} \frac{r_o}{D} &= .67\%; \; V = 150\text{cm/s} && (31) \\ &= .74\%; \; V = 100\text{cm/s} && (32) \end{aligned}$$

References

[1] George R. Cooper and Clare D. McGillem. *Probabilistic Methods of Signal and System Analysis.* Holt, Rinehart and Winston, 1971.

[2] Athanasios Papoulis. *Probability, Random Variables, and Stochastic Processes.* McGraw-Hill, 1965.