

INTRODUCTION

This book represents an alternative approach to that found in most standard texts to the study of image formation by spherical surfaces. As such it complements the usual treatment and may well be used along with another text. The text "Fundamentals of Optics" by F. A. Jenkins and H. E. White (Toronto, McGraw-Hill, 3rd ed., 1957) has been referred to throughout as it uses a sign convention which is consistent with that used in this book.

The subject of geometrical optics starts with the laws of refraction and reflection for transparent media. It is then a question of using these laws to discover the properties of various optical systems which may contain any number of curved refracting and reflecting surfaces. In principle we could solve any optical problem by the exact application of the basic laws. However, it is possible to derive some very useful general results by using approximate forms of the laws which treat only rays, called paraxial rays, which make small angles with respect to an optic axis. Also, it is usual to confine the treatment to spherical surfaces. This approximate form of the basic laws as applied to paraxial rays in an optical system consisting of spherical surfaces is called the first order theory and constitutes a major part of the subject of geometrical optics.

The first order theory forms the basis for obtaining the equation describing image formation which relates image position to object position and defines magnification for the case of a single spherical refracting surface between two transparent media. For an extensive optical system this equation can be applied repeatedly to each surface of the system in turn. However the equations so derived quickly become involved algebraically for all but the simplest cases, such as, for example, the single thin lens in air.

The matrix method is another approach to the study of optical systems using the first order theory. By properly specifying the rays passing through the optical system it is possible to reduce a problem to a succession of matrix operations. This not only simplifies the algebraic manipulation but more importantly reveals some very general properties of optical systems not easily obtainable in other ways.

From the mathematical point of view, the first order theory can be treated in terms of linear transformations. And, since matrix algebra is the most suitable way of handling linear transformations, it is obvious that we should apply the results of matrix algebra to treat the first order optical problem.

The matrix method then represents a basic approach to the analysis of optical systems which results in discovering the cardinal points of a lens or mirror system from a more general point of view. The familiar lens equations for an optical system also result in both the Newtonian and Gaussian forms. This much is accomplished part way through chapter 5. Included in this is a chapter which discusses the properties of two by two matrices as they describe the linear transformation involved in an optical system. The rest of chapter 5 shows a number of alternative ways of expressing image formation in terms of different cardinal points of a lens system. In chapter 6 some practical optical systems are discussed in terms of the matrix approach.

Chapter 7 is a discussion of spherical mirrors by the matrix method. In the Appendix is given a simple computer programme for calculating the matrix for a lens system and some of the characteristics of the system.

Chapter 1

Physical Basis of Geometrical Optics

As mentioned, the subject of geometrical optics is based on the laws of reflection and refraction. The laws of reflection state:

1. The reflected ray lies in the plane of incidence.
2. The angle of reflection equals the angle of incidence.

The plane of incidence is the plane containing the incident ray and the normal to the reflecting surface constructed at the point where the incident ray strikes the surface. The angle of reflection is the angle between the normal and the reflected ray; the angle of incidence is the angle between the normal and the incident ray.

The laws of refraction state that for a refracting surface between two transparent media:

1. The refracted (or transmitted) ray lies in the plane of incidence, and
2. $n \sin \theta = n' \sin \theta'$, (Snell's law),

where θ is the angle of incidence and θ' is the angle of refraction, the angle between the normal to the surface and the refracted ray, and n and n' are the indices of refraction of the media on either side of the refracting surface. The index of refraction of a medium is defined as the ratio of the speed of light in vacuum to the speed of light in the medium.

In first order theory, which essentially involves the first term in the series expansion for the sine of the angle, Snell's law is used in the form $n \theta = n' \theta'$. A better approximation than the first order theory involves higher order terms in the expansion which gives rise to the third and fifth order theories. The results of the higher order theories are usually expressed in terms of the defects in the image formed by spherical refracting surfaces, the so-called monochromatic aberrations.

In what follows we shall concentrate on the subject of refraction, on the basis of first order theory, leaving the subject of reflection to be treated later as a special case.

Chapter 2

Basic Matrix Algebra

Linear Transformations and Matrix Manipulation

The geometrical laws in terms of the first order theory may be stated in the form of linear transformations which in turn are most easily treated by matrix algebra. In particular we may represent the physical situation in terms of two linear equations. The two variables we shall call y and α . The physical meaning attached to these two variables will be stated later. We will be able to show that two other variables y' and α' are related to y and α through two linear equations of the form:

$$y = b_{11}y' + b_{12}\alpha', \quad (1)$$

$$\alpha = b_{21}y' + b_{22}\alpha', \quad (2)$$

where b_{11} , b_{12} , b_{21} , and b_{22} are constants, which we will later show to be characteristic of a given optical system. These constants may be written as the elements of a 2 x 2 matrix,

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

The linear equations (1) and (2) can be written as the matrix equation

$$\begin{bmatrix} y \\ \alpha \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}. \quad (3)$$

In fact, equations (1) and (2) constitute a definition of equation (3).

Now suppose that a further linear transformation is known which connects the variables y'' and α'' say, to y' and α' such that

$$y' = c_{11}y'' + c_{12}\alpha'' \quad (4)$$

$$\alpha' = c_{21}y'' + c_{22}\alpha'', \quad (5)$$

or in matrix notation

$$\begin{bmatrix} y' \\ \alpha' \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} y'' \\ \alpha'' \end{bmatrix}. \quad (6)$$

We can discover the equations which relate y and α to y'' and α'' by substituting equations (4) and (5) into (1) and (2). Then we have

$$y = b_{11}(c_{11}y'' + c_{12}\alpha'') + b_{12}(c_{21}y'' + c_{22}\alpha''),$$

$$\alpha = b_{21}(c_{11}y'' + c_{12}\alpha'') + b_{22}(c_{21}y'' + c_{22}\alpha''),$$

or

$$y = (b_{11}c_{11} + b_{12}c_{21})y'' + (b_{11}c_{12} + b_{12}c_{22})\alpha'', \quad (7)$$

$$\alpha = (b_{21}c_{11} + b_{22}c_{21})y'' + (b_{21}c_{12} + b_{22}c_{22})\alpha'', \quad (8)$$

which is a new transformation of the form

$$y = a_{11}y'' + a_{12}\alpha'', \quad (9)$$

$$\alpha = a_{21}y'' + a_{22}\alpha''. \quad (10)$$

In matrix notation,

$$\begin{bmatrix} y \\ \alpha \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y'' \\ \alpha'' \end{bmatrix} \quad (11)$$

where the " a " coefficients are defined by comparing equations (9) and (10) with equations (7) and (8).

The matrix formulation involves the substitution of the expression for the column matrix

$$\begin{bmatrix} y' \\ \alpha' \end{bmatrix}$$

of equation (6) into equation (3) to give

$$\begin{bmatrix} y \\ \alpha \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} y'' \\ \alpha'' \end{bmatrix}. \quad (12)$$

Thus comparing equations (11) and (12) we have

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}. \quad (13)$$

This essentially defines the usual rule of matrix multiplication given by the coefficients of equation (7) and (8) as:

$$a_{11} = b_{11}c_{11} + b_{12}c_{21}$$

$$a_{12} = b_{11}c_{12} + b_{12}c_{22}$$

$$a_{21} = b_{21}c_{11} + b_{22}c_{21}$$

$$a_{22} = b_{21}c_{12} + b_{22}c_{22}$$

or more compactly,

$$a_{ik} = \sum_{j=1}^2 b_{ij}c_{jk}. \quad (14)$$

The matrix equation may be written as

$$[\mathbf{A}] = [\mathbf{B}][\mathbf{C}].$$

Exercises

1. Show that

$$\begin{bmatrix} 6 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 30 & 27 \\ 8 & 3 \end{bmatrix}.$$

2. Show that

$$\begin{bmatrix} 4 & 5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 6 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & 22 \\ 11 & 4 \end{bmatrix}.$$

3. Show that in general

$$[\mathbf{A}][\mathbf{B}] \neq [\mathbf{B}][\mathbf{A}].$$

4. For what particular matrix $\mathbf{B}$ does

$$[\mathbf{A}][\mathbf{B}] = [\mathbf{B}][\mathbf{A}]?$$

5. Show that

$$\begin{bmatrix} 6 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 45 \\ 5 & 37 \end{bmatrix}.$$

6. Show that

$$\begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 41 & 17 \\ 5 & 2 \end{bmatrix}.$$

7. Show that the multiplication by the matrix

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

and any matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

always leaves the element a_{21} unchanged.

8. Compute the product matrix

$$\begin{bmatrix} 1 & 5 \\ 7 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 7 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 8 & 3 \\ 1 & 7 \end{bmatrix}.$$

Determinant of a Matrix

We shall have occasion to make use of one important property of matrices which involves the determinant of a matrix.

The determinant of the matrix $[A]$ is written as

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

and is defined as the number $(a_{11} a_{22} - a_{21} a_{12})$.

The theorem that we are interested in states that the determinant of the product of a number of matrices is equal to the product of the determinants of each of the matrices forming the product. We leave the demonstration of this theorem to the exercises below.

Exercises

9. Show that

$$\begin{vmatrix} 6 & 3 \\ 1 & 2 \end{vmatrix} \begin{vmatrix} 4 & 5 \\ 2 & -1 \end{vmatrix} = \begin{vmatrix} 30 & 27 \\ 8 & 3 \end{vmatrix} = -126.$$

10. Show that

$$\begin{vmatrix} 4 & 5 \\ 2 & -1 \end{vmatrix} \begin{vmatrix} 6 & 3 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 29 & 22 \\ 11 & 4 \end{vmatrix} = -126.$$

11. Show that if $[C] = [B][A]$ then $|B||A| = |A||B| = |C|$.

12. Extend the result of exercise 11 to show that for N matrices such that

$$[P] = [A_1][A_2] \dots [A_n],$$

$$|P| = |A_1||A_2| \dots |A_n|.$$

13. Show that the result of exercise 12 holds for the product matrix of exercise 8 above.

We are not going to be concerned with other properties of matrices. The interested student may find these in any standard text on matrix algebra.

Chapter 3

The Basic Matrix Transformations of Geometrical Optics

In this chapter we shall first define the variables y and α that were used in the preceding pages. Then we shall derive two different linear transformations, one corresponding to a translation, the other to a refraction operation. They will be characterized by a translation matrix and a refraction matrix respectively. With these two matrices we shall be able to treat any optical system according to the first order theory.

1. The Variables

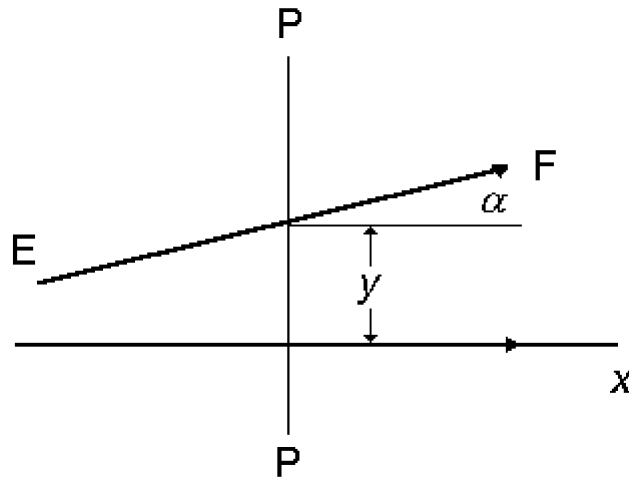


Figure 1

Let us consider a ray of light EF as in Fig. 1 which is traversing a uniform portion of a transparent medium which forms part of an optical system. The ray makes a small angle with respect to the x direction which we will designate as the optic axis of the system. The centres of curvature of all the spherical refracting surfaces lie on the optic axis. We shall choose a plane PP which is perpendicular to the optic axis.

The meaning of the variables y and α can be seen from Fig. 1. For a given plane PP the variable y represents the height above the x axis that the ray strikes the plane PP . The variable α represents the angle the ray makes with the x axis and since the slope $(dy/dx) = \tan(\alpha) = \alpha$ for small angles, we may take α as representing the slope of the ray as it passes the plane PP . The distance y above the plane is positive, below the plane negative. The sign of the angle α is determined from the sign of the slope and is positive in Fig. 1.

2. The Translation Matrix

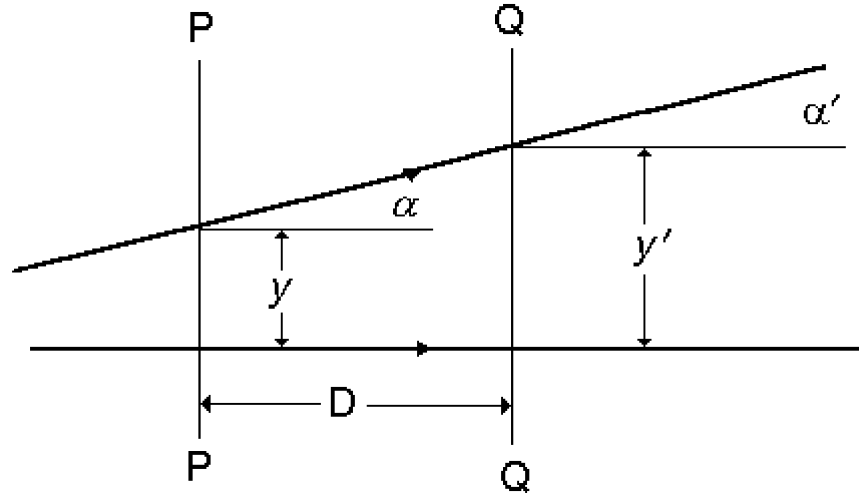


Figure 2

In Fig. 2 the ray EF is progressing (without change in direction) through the uniform medium as described for Fig. 1. We shall always draw the ray as progressing from left to right in the x direction. We may designate two planes, one PP and the other QQ, a distance D from PP; the ray is characterised by the variables y, α , at PP and y', α' , at QQ. From Fig. 2 we may write

$$y = y' - D\alpha',$$

(for small angles) and since α' equals α we may write another linear equation as

$$\alpha = 0y' + \alpha'.$$

These linear transformation equations can be written as

$$\begin{bmatrix} y \\ \alpha \end{bmatrix} = \begin{bmatrix} 1 & -D \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}. \quad (15)$$

The matrix,

$$\begin{bmatrix} 1 & -D \\ 0 & 1 \end{bmatrix}$$

is known as the translation matrix.

A series of translations may be represented according to the rules of matrix multiplication already discussed. For example, suppose that the ray strikes a further plane WW at a distance T from QQ at angle α'' and height y'' . In matrix form we could write

$$\begin{bmatrix} y \\ \alpha \end{bmatrix} = \begin{bmatrix} 1 & -D \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y'' \\ \alpha'' \end{bmatrix} = \begin{bmatrix} 1 & -(D+T) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y'' \\ \alpha'' \end{bmatrix}.$$

That is,

$$y = y'' - (T + D)\alpha'',$$

$$\alpha = 0 + \alpha''$$

which can be verified by examining Fig. 3.

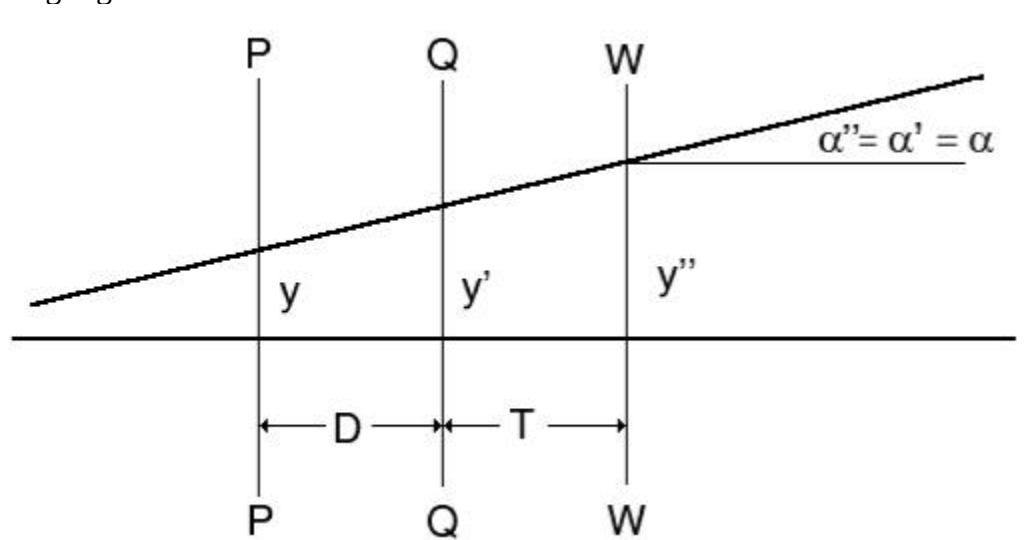


Figure 3.

The distances T and D are always taken as positive quantities as the transformation is defined in terms of the ray direction. With the convention chosen here for expressing the unprimed coordinates as functions of the primed coordinates as in equation 15, the order of the matrices is in the direction of the ongoing ray, i.e. from left to right.

3. The Refraction Matrix

At each surface separating two transparent media within an optical system, the direction of the ray may be changed according to Snell's law, which we take in the approximate form for small angles. We may express this refraction at the surface in terms of two linear equations which gives rise to the refraction matrix.

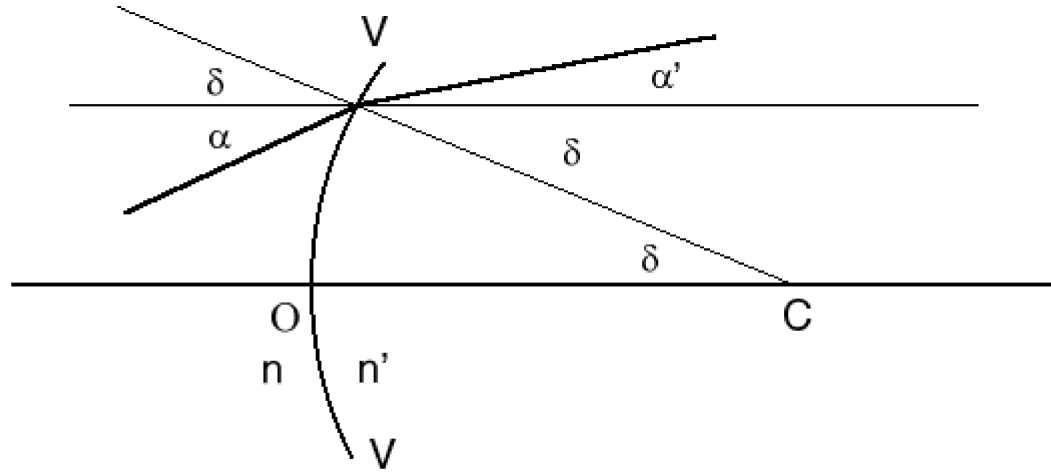


Figure 4.

In Fig. 4, VV represents the spherical refracting surface separating transparent media of indices n and n' respectively. The intersection of the surface and the optic axis is the vertex O. The radius of curvature is R and is considered positive (as in the figure) if the direction OC is the positive x direction where C is the centre of curvature of the surface. Since we have confined the discussion to small angles and rays close to the axis, the variables y and α may be considered to refer to the plane through O perpendicular to the optic axis.

At the surface the angle α changes from α to α' while the variable y is simply the height above the x axis of the point at which the ray strikes the surface and refraction takes place. Thus we can write either y or y' for this height and the transformation equation may be written as

$$y = y' + 0\alpha'. \quad (16)$$

We may now derive the transformation equation for α at this surface using Snell's law. From Fig. 4,

$$n\theta = n'\theta' \quad (\text{approximately})$$

or

$$n(\delta + \alpha) = n'(\delta + \alpha')$$

or

$$n'\alpha' = n\alpha + (n - n')\delta$$

and

$$\delta = y'/R.$$

Therefore we obtain

$$n\alpha = n'\alpha' + (n - n')y'/R. \quad (17)$$

Thus the transformation equations can be written for the plane through O using equation (16) and rearranging (17) as,

$$y = y' + 0\alpha', \quad (18)$$

$$\alpha = \left(\frac{n' - n}{nR}\right)y' + \left(\frac{n'}{n}\right)\alpha', \quad (19)$$

or in matrix form

$$\begin{bmatrix} y \\ \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n' - n}{nR} & \frac{n'}{n} \end{bmatrix} \begin{bmatrix} y' \\ \alpha' \end{bmatrix} \quad (20)$$

and the matrix,

$$\begin{bmatrix} 1 & 0 \\ \frac{n' - n}{nR} & \frac{n'}{n} \end{bmatrix}$$

is defined as the refraction matrix for the plane through the vertex O of the refracting surface.

The Basic Matrix Transformations of Geometrical Optics

Translation and Refraction

Now we have two basic matrix operations which describe the progress of a ray through an optical system. We need only apply the laws of matrix multiplication to determine the transformation between any two planes using the translation and refraction matrices. We illustrate this with the following example:

A glass rod of index $n' = 1.5$ has a hemispherical convex surface of radius 6 cm on one end as in Fig. 5. A ray makes an angle α equal to 0.1 radians with the plane PP at a height of 1.6 cm from the optic axis. The plane PP is 8 cm to the left of the vertex O. Find the height of the ray y' and its direction α' at the plane QQ 9 cm to the right of O.

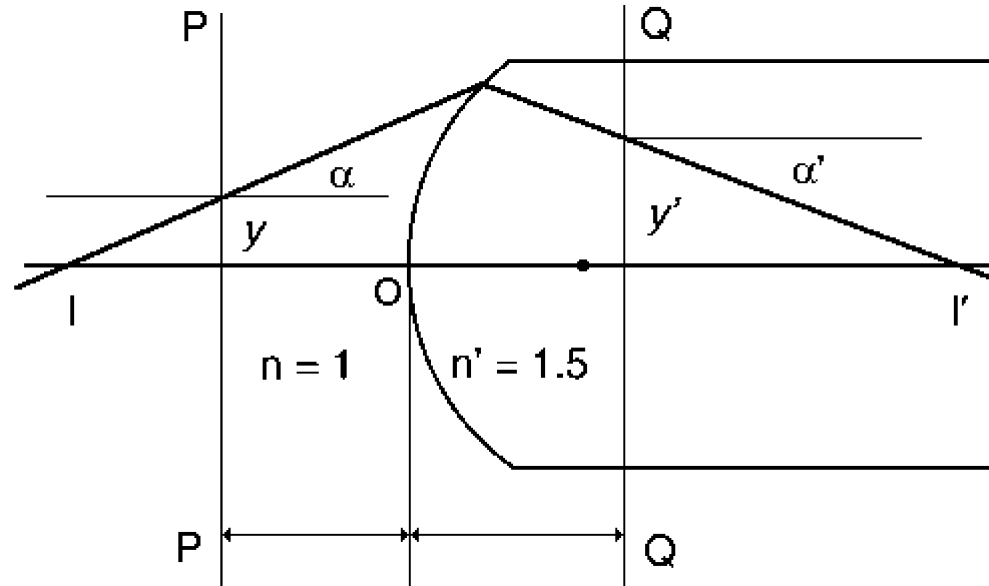


Figure 5

Solution:

$$\begin{bmatrix} y \\ \alpha \end{bmatrix} = \begin{bmatrix} 1 & -8 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1.5-1}{6} & 1.5 \end{bmatrix} \begin{bmatrix} 1 & -9 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}$$

or

$$\begin{bmatrix} y \\ \alpha \end{bmatrix} = \begin{bmatrix} 1/3 & -15 \\ 1/12 & 3/4 \end{bmatrix}_{PQ} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}.$$

Note that we can check that the determinant of the product matrix is correct since

$$\begin{vmatrix} 1/3 & -15 \\ 1/12 & 3/4 \end{vmatrix}_{PQ} = \begin{vmatrix} 1 & -8 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 1/12 & 3/2 \end{vmatrix} \begin{vmatrix} 1 & -9 \\ 0 & 1 \end{vmatrix} = \frac{3}{2}.$$

Now

$$\begin{bmatrix} 1.6 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 1/3 & -15 \\ 1/12 & 3/4 \end{bmatrix} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}$$

or

$$\begin{aligned} 1.6 &= (1/3)y' - 15\alpha' \\ 0.1 &= (1/12)y' + (3/4)\alpha'. \end{aligned}$$

The solution to these equations is $y' = 1.8$ cm and $\alpha' = -.0667$ radians.

It is perhaps well to emphasize at this point that we have not yet mentioned the idea of image formation and y' of the example is certainly not the image of y , since a ray originating at y will go through the plane QQ at a height y' which depends on α . If α were .05 radians say, in the last example then a ray through $y = 1.6$ cm passes the plane QQ at the height $y' = 1.3$ cm instead of 1.8 cm. This is illustrated in Fig. 6.

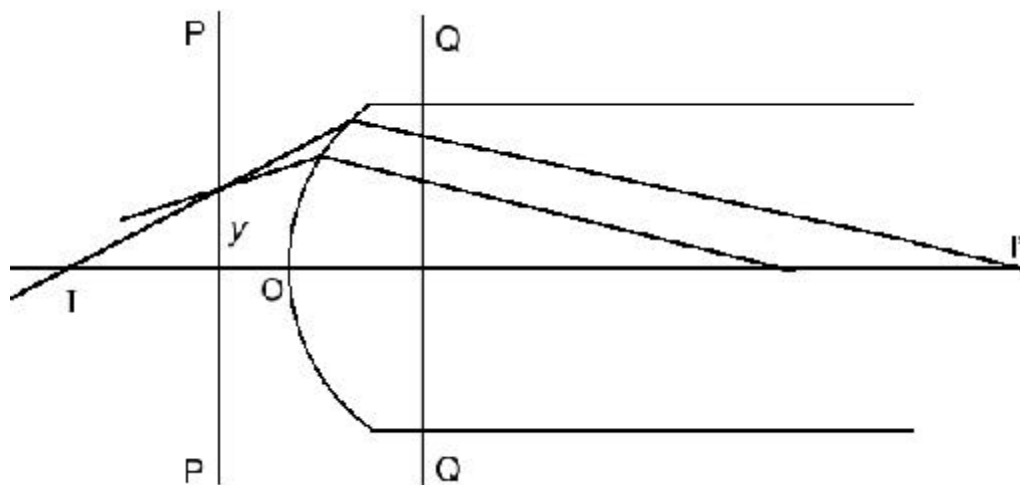


Figure 6.

In anticipation of further sections, we can observe that I' in Fig. 5 appears to be an image of I as the ray along the optic axis (which is undeviated) and the ray at angle α both originate at I and pass through I' . To prove that I' is indeed an image of I we need to show that all the rays from I pass through I' regardless of the angle α as in Fig. 7. We choose the planes through I and I' as our reference planes and illustrate this with the help of the following exercise.

Exercise 1

The numbers in this example have been chosen so that I' is 36 cm to the right of O and I is 24 cm to the left of O .

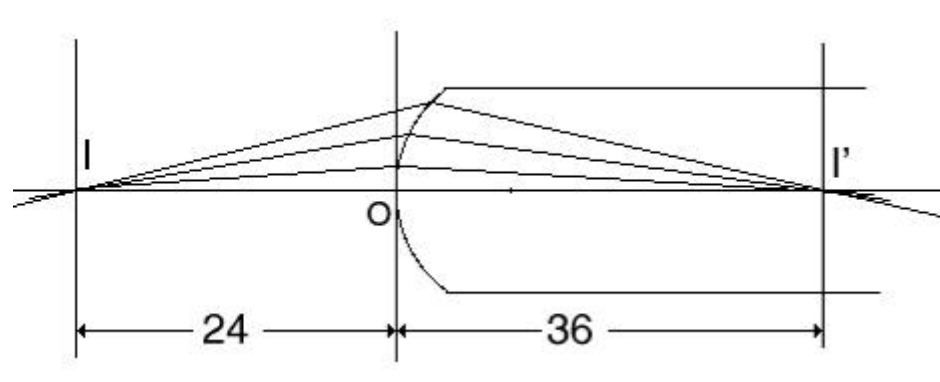


Figure 7.

Find the matrix connecting the planes through I and I' and show that

$$\begin{bmatrix} y \\ \alpha \end{bmatrix}_I = \begin{bmatrix} -1 & 0 \\ 1/12 & -3/2 \end{bmatrix}_{II'} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}_{I'}$$

where now y' , α' , represents the ray at the I' plane and y , α , the ray at the I plane.

In this exercise we note that $y' = -y$ independent of α so that if $y = 0$ (point I) then $y' = 0$ as illustrated in Fig. 7 which shows I' to be an image of I .

As can be seen from the above exercise, if the matrix element in the upper right hand corner a_{12} is zero, then the two planes are such that y' depends only on y . With this fact we can derive the usual equation (equation (21) below) for image formation by refraction at a spherical surface.

Exercise 2

Let s be the distance OI and s' be OI' . If the index on the left is n , on the right n' and R is the radius of curvature, then

$$\begin{bmatrix} y \\ \alpha \end{bmatrix}_s = \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (n' - n)/nr & n'/n \end{bmatrix} \begin{bmatrix} 1 & -s' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}.$$

Show that:

$$\frac{n'}{s'} + \frac{n}{s} = \frac{n' - n}{r} \quad (21)$$

if I' is an image of I .

Sign Convention

Equation (21) does presuppose the sign convention of Jenkins and White, for example, in which the object distance s is positive if the object is to the **LEFT** of O and the image distance s' is positive if the image is to the **RIGHT** of O .

In the above example we have used positive image and object distances. In Figs. 8 and 9 we illustrate negative distances which in these cases represent a virtual object and a virtual image.

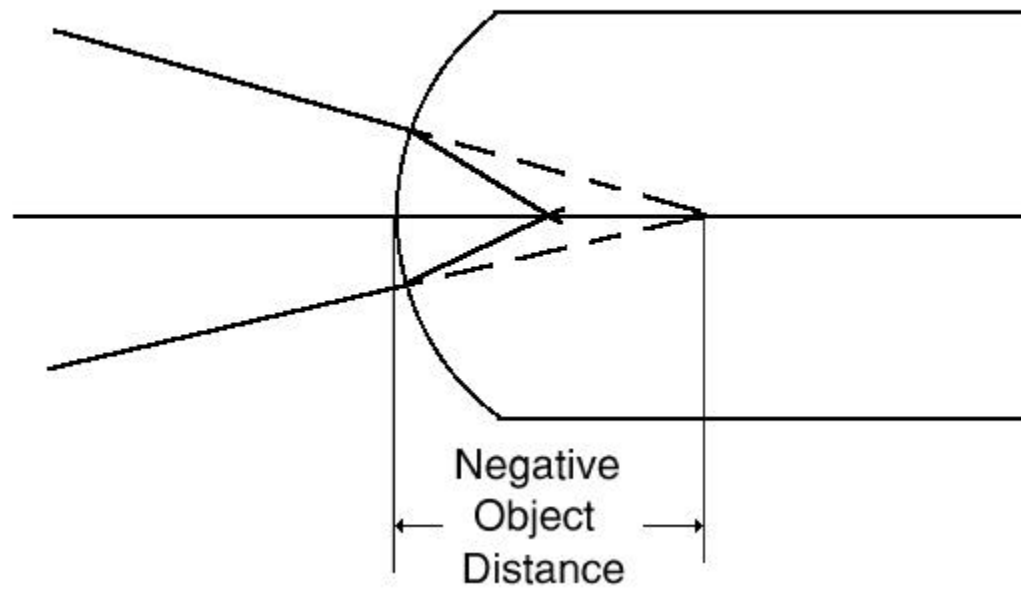


Figure 8.

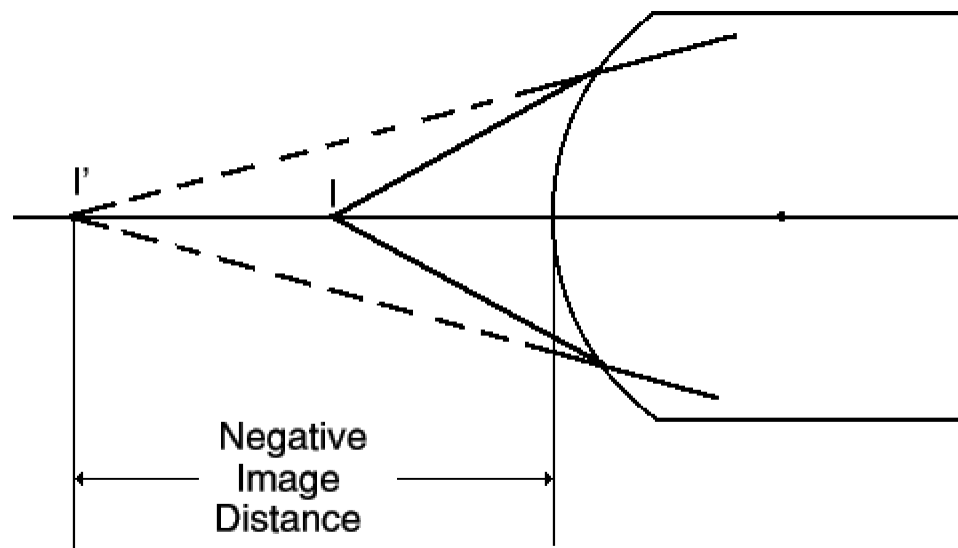


Figure 9.

Generally we shall be able to extend the sign convention to a complex optical system in which the relevant distances are measured from various cardinal points which we shall define for the system. The cardinal points are readily obtained using the matrix approach.

Chapter 4

The General Optical System

The General Matrix

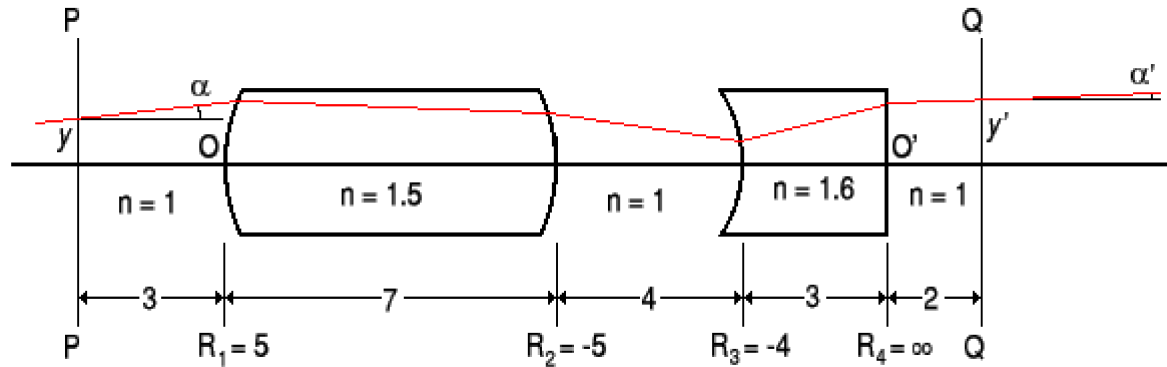


Figure 10.

In the preceding discussion we were able to describe the passage of a ray from one point to another in a simple optical system by means of two translation matrices and one refraction matrix. A more complicated optical system may contain many refracting surfaces of different indices of refraction. Let us for example consider the system of Fig. 10. A ray passes the plane PP at height y , angle α .

$$\begin{bmatrix} y \\ \alpha \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ .1 & 1.5 \end{bmatrix} \begin{bmatrix} 1 & -7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ .0667 & .667 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.15 & 1.6 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .625 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}.$$

These nine matrices may be multiplied together to give one two by two matrix relating $y' \alpha'$ and $y \alpha$ at the planes QQ and PP respectively. If the plane PP is moved to the vertex O and QQ to O' then we have the matrix,

$$\begin{bmatrix} 1 & 0 \\ .1 & 1.5 \end{bmatrix} \begin{bmatrix} 1 & -7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ .0667 & .667 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.15 & 1.6 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .625 \end{bmatrix},$$

which represents the effect on a ray in going from the plane through the first vertex O of the system through to the plane through the final

vertex O ' of the system. It is useful in describing the general optical system to base our discussion on this matrix which we will designate as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{OO'}$$

which in this example equals

$$\begin{bmatrix} 1.554 & -0.713 \\ 0.1654 & -0.390 \end{bmatrix}.$$

Obtaining the elements of the matrix may involve a lengthy series of matrix multiplications as in our example above, involving 7 matrices. However, all the necessary information about the optical system is included in this resultant two by two matrix and from it the cardinal points and planes of an optical system are readily obtainable. From these cardinal points we may completely describe the image-forming characteristics of the system.

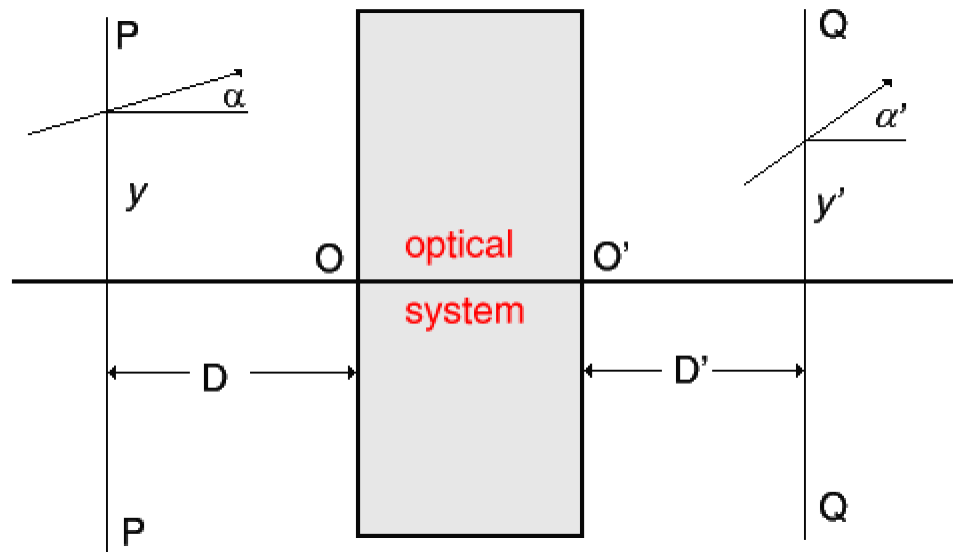


Figure 11.

In general then, we can assume the situation of Fig. 11. We know (or can compute) the matrix relating the rays going from the plane through O to the plane through O ' namely

$$[\mathbf{A}]_{OO'} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{OO'}.$$

The matrix equation relating the emergent ray ($y' \alpha'$) at plane QQ a distance D' from O' to the incident ray ($y \alpha$) at the plane PP a distance D to the left of O is

$$\begin{bmatrix} y \\ \alpha \end{bmatrix}_P = \begin{bmatrix} 1 & -D \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{OO'} \begin{bmatrix} 1 & -D' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}_Q$$

or,

$$\begin{bmatrix} y \\ \alpha \end{bmatrix}_P = \begin{bmatrix} a_{11} - Da_{21} & D'(-a_{11} + Da_{21}) + a_{12} - Da_{22} \\ a_{21} & a_{22} - D'a_{21} \end{bmatrix}_{PQ} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}_Q. \quad (22)$$

In Fig. 10 we had $D' = 2$ cm and $D = 3$ cm.

We are now in a position to discuss the first two cardinal points (and planes), the first and second focal points (and planes).

Chapter 4

The General Optical System

The Focal Points and Planes

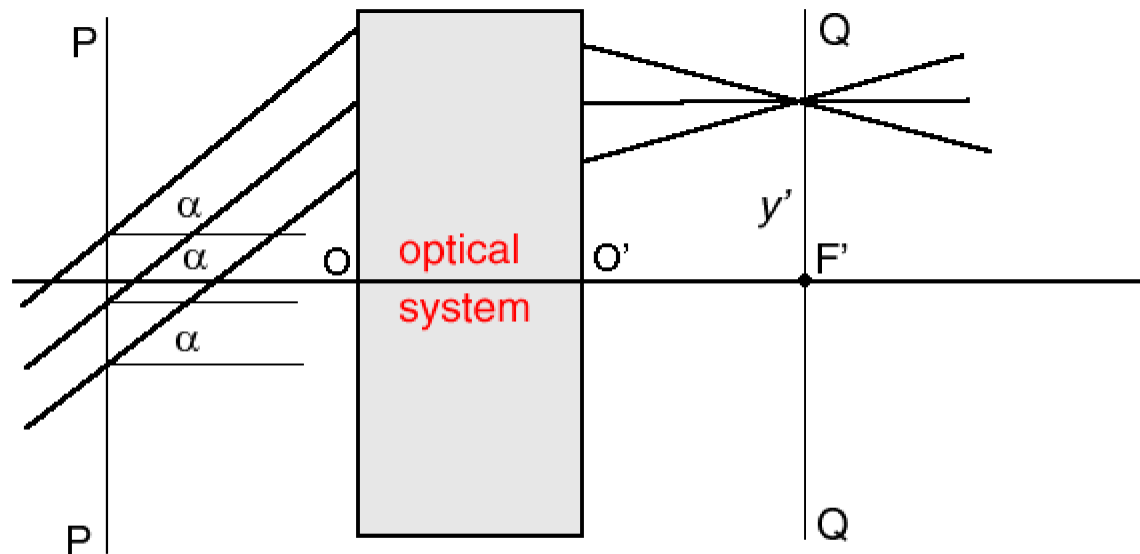


Figure 12.

If a beam of parallel rays enters an optical system the emergent beams will converge toward a point (or diverge away from a point for a negative system). This point will lie in the second focal plane. If the rays are parallel to the optic axis of a system the point is the second focal point. The focal plane passes through the focal point and is perpendicular to the optic axis. This is illustrated in Fig. 12.

We shall now determine the position of the second focal point, F' , in terms of the elements of the matrix $[A]_{OO'}$. To do this let us take our second reference plane QQ to be the second focal plane in Fig. 12. Then $D' = O'F'$. The first reference plane PP can be anywhere to the left of O. We now note that every ray of the parallel beam passing through PP no matter what its y coordinate is, passes through the same point y' in the focal plane. That is, one of the linear equations relating y', α' to y, α must be of the form

$$y' = (\text{constant})\alpha \quad (\text{independent of } \alpha').$$

Thus, the lower diagonal element of the matrix of equation (22) must be zero so that,

$$D' = O'F' = a_{22}/a_{21} \quad (23)$$

and thus

$$\alpha = a_{21}y'.$$

The position of the second focal point, F' , relative to the second vertex O' is thus determined in terms of the matrix elements relating O' to O . A positive value of D' indicates that F' is to the right of O' ; a negative value means F' is to the left of O' .

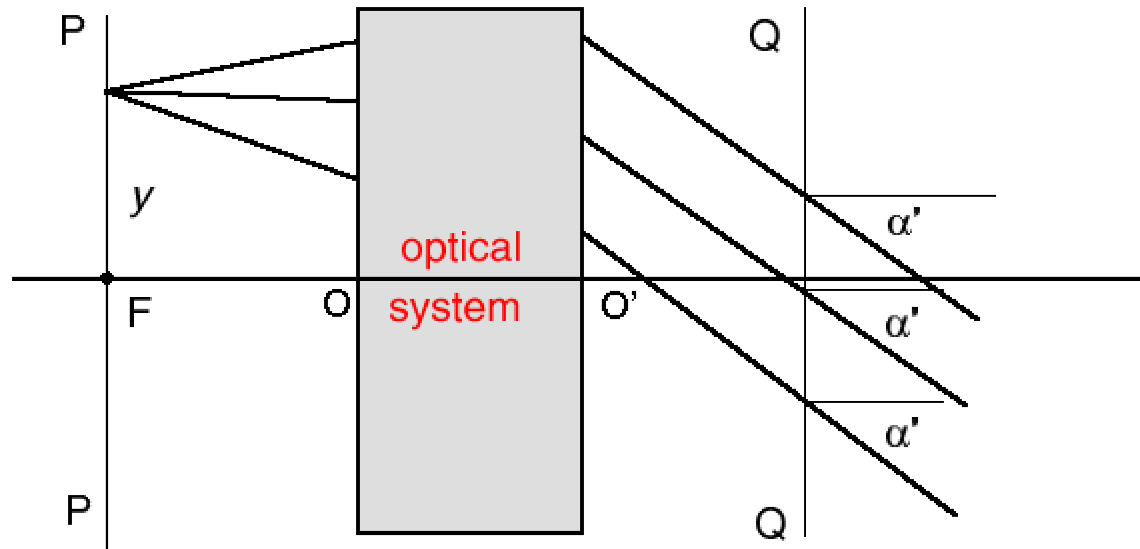


Figure 13.

Rays emerging from a point in the first focal plane are illustrated in Fig. 13. A parallel beam emerges from the optical system. We can take the first reference plane PP through F and QQ the second reference plane anywhere to the right of O' . Now we note that the rays through Q' pass this plane at constant angle α' although the incident rays all have different angles α at the plane PP, the focal plane. So one of the linear equations must state that α' is independent of α , (and depends only on y). Equation (22) gives this relation if the upper diagonal element is zero, that is, if

$$D = OF = a_{11}/a_{21}. \quad (24)$$

Here D is positive if F is to the left of O, negative if F is to the right of O. Then we have from equation (22) that

$$y = (a_{12} - Da_{22})\alpha'. \quad (25)$$

Thus the focal points may be easily found from (23) and (24) once the matrix $[A]_{OO'}$ is computed.

The Matrix Relating the Focal Planes

It is extremely useful to choose the reference planes PP and QQ to be the first and second focal planes respectively. The matrix equation relating the emergent ray y', α' , at the second focal plane to an incident ray y, α , at the first focal plane is given by

$$\begin{bmatrix} y \\ \alpha \end{bmatrix}_F = [\text{matrix}]_{FF'} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}_{F'}.$$

From the discussion above, the matrix has a particularly simple form, since both diagonal elements are zero. Using (23) and (24) in (22), we obtain

$$\begin{bmatrix} y \\ \alpha \end{bmatrix}_F = \begin{bmatrix} 0 & (a_{12} - Da_{22}) \\ a_{21} & 0 \end{bmatrix}_{FF'} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}_{F'}. \quad (26)$$

The significance of the two non-zero elements of the matrix connecting the two focal planes is not evident at this point. However, as will be shown later the elements are related to the first and second focal lengths, f and f' , defined as the distances from the first and second principal points to the first and second focal points respectively (see Fig. 17). The principal points are defined later as well. In the familiar case of the thin lens, the two principal points are coincident at the lens centre. For the moment we can anticipate the later results by designating the two non-zero matrix elements of equation (26) as

$$a_{21} = 1/f \quad (28)$$

and

$$(a_{12} - Da_{22}) = -f'. \quad (29)$$

A very general relationship between f and f' may be easily shown by using the theorem on the product of determinants of matrices. First let us note that the determinant of a translation matrix,

$$\begin{bmatrix} 1 & -D \\ 0 & 1 \end{bmatrix},$$

is unity and the determinant of the refraction matrix separating two media of index n_1 and n_2 say, is

$$\begin{bmatrix} 1 & 0 \\ (n_2 - n_1)/n_1 R & n_2/n_1 \end{bmatrix} = n_2/n_1.$$

Now if there is a series of refracting surfaces separating media with indices of refraction $n, n_1, n_2, \dots, n'$, starting with the medium of index n on the left and ending with the medium n' on the right then the determinant of the product matrix is given by:

$$(n/n_1)(n_1/n_2)(n_2/n_3) \dots (n_i/n_{i-1})(n'/n_i) = n'/n,$$

ignoring all the translations as they involve multiplication by one. For the optical system $|A|_{OO'} = n'/n$, and since a translation from the planes through the vertices O and O' of the optical system to the first and second focal planes respectively does not change the value of the determinant then

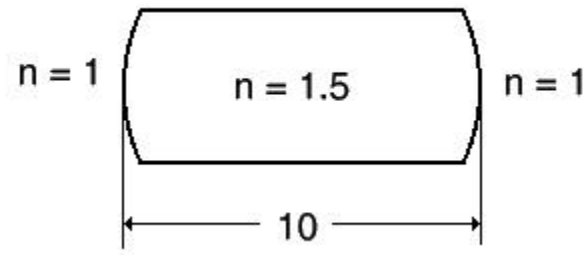
$$\begin{vmatrix} 0 & a_{12} - Da_{22} \\ a_{21} & 0 \end{vmatrix} = \begin{vmatrix} 0 & -f \\ 1/f & 0 \end{vmatrix} = f'/f = n'/n \quad (30)$$

The ratio of the first and second focal lengths is seen to be equal to the ratio of the index on the left of O to the index on the right of O'. Also f and f' must have the same sign.

Equation (30) illustrates the power of the matrix method as the result is difficult to demonstrate in other ways for even a single thick lens and a general proof is not usually attempted for more complicated systems.

It is also perhaps well to note that the theorem on matrices constitutes a useful check on the accuracy of multiplication in computing $[A]_{OO'}$. In the example of Fig. 10, the final product matrix (involving 7 matrices) should have a determinant equal to unity since we had $n = n' = 1$.

Example



A thick biconvex lens in air has a radius of curvature of 10 cm at each surface and the surfaces are 10 cm apart. Find the focal points and the focal lengths of the lens if the index of the glass of the lens is 1.50.

Solution:

$$[A]_{OO'} = \begin{bmatrix} 1 & 0 \\ (1.5 - 1)/10 & 1.5/1 \end{bmatrix} \begin{bmatrix} 1 & -10 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (1 - 1.5)/1.5(-10) & 1/1.5 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & -20/3 \\ 5/60 & 2/3 \end{bmatrix}$$

$$f = 60/5 = 12 \text{ cm} = f' \text{ since } n = n' = 1.$$

$$OF = (2/3) \times (12) = 8 \text{ cm. } O'F' = (2/3) \times (12) = 8 \text{ cm.}$$

F is 8 cm to the left of O, F' is 8 cm to the right of O'.



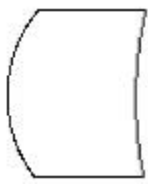
Exercise 1

Take the lens of the example to be 1 cm thick and show that $f = 10.17 \text{ cm}$, $OF = 9.83 \text{ cm}$, $O'F' = 9.83 \text{ cm}$.



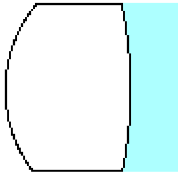
Exercise 2

Take the lens to be a biconcave thick lens with radii of 10 cm, index 1.5, thickness 1 cm. Show that $f = f' = -9.84 \text{ cm}$, $OF = -10.16 \text{ cm}$, $O'F' = -10.16 \text{ cm}$.



Exercise 3

A thick lens is concavo convex, the first radius is 10 cm, the second radius is 20 cm. If the lens is in air and has an index = 1.5 and the thickness = 3 cm, show that $f' = f = 36.36$ cm,



Exercise 4

A biconvex lens of index 1.50 has radii 10 cm and 20 cm, a thickness of 10 cm. Air is to the left of O and water (of index $4/3$) is to the right of O'. Show that $f' = 24$ cm, $f = 18$ cm, O 'F' = 16 cm, OF = 17 cm.

Exercise 5

Find the matrix $[A]_{OO'}$, and the $[matrix]_{FF'}$, for a thin lens (that is, the thickness of the lens is taken to be zero), with index n on the left, n' on the right and the index of the lens, μ . Take the radii of curvature as R_1 and R_2 .

The above exercises may be worked using the computer programme in the [appendix](#).

Chapter 5

Image Formation

Newton's Equations

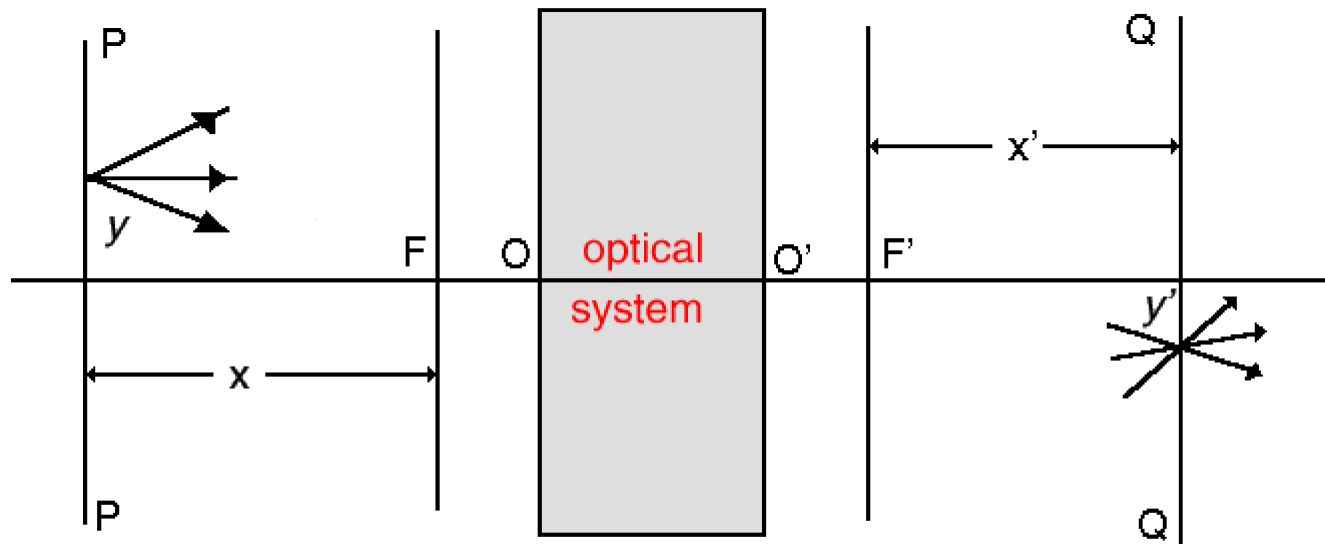


Figure 14.

In Fig. 14 is illustrated the formation of an image point at y' of an object point at y by an optical system between O and O' . The plane PP represents the plane through the object point, a distance x to the left of O ; QQ is the plane through the image point, a distance x' to the right of O' . The focal points F and F' can be determined as discussed in the previous chapter and the matrix relating a ray passing from the first focal plane to the second is given by

$$\begin{bmatrix} 0 & -f' \\ 1/f & 0 \end{bmatrix}_{FF'}.$$

The planes PP and QQ can be related by using this matrix and two further translation matrices. Thus a ray y, α , at PP emerges as the ray y' ,

α' , at QQ such that

$$\begin{bmatrix} y \\ \alpha \end{bmatrix}_P = \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix}_{PF} \begin{bmatrix} 0 & -f' \\ 1/f & 0 \end{bmatrix}_{FF'} \begin{bmatrix} 1 & -x' \\ 0 & 1 \end{bmatrix}_{F'Q} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}_Q \quad (31)$$

or

$$\begin{bmatrix} y \\ \alpha \end{bmatrix}_P = \begin{bmatrix} -x/f & x'x/f - f' \\ 1/f & -x'/f \end{bmatrix}_{PQ} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}_Q. \quad (32)$$

Since all the rays emerging from y arrive at y' regardless of the angle α , the upper right hand element of the matrix of (32) must be 0, in which case

$$xx' = ff' \quad (33)$$

and

$$y = -(x/f)y' \text{ (independent of } \alpha \text{)}. \quad (34)$$

It is clear from Fig. 14 that an object of height y is imaged as an image of height y'. The lateral magnification, m, is defined as the ratio y'/y which from equation (34) is given by

$$m = y'/y = -f/x = -x'/f'. \quad (35)$$

Equations (33) and (35) are Newton's equations for image formation and apply generally to any optical system.

Example 1

An object 2 cm high is placed 40 cm to the left of the vertex O of the thick lens of the [example of chapter 4](#). Find the image position and image size.

Solution: Since F is 8 cm to the left of O, $x = 32$ cm

$$32x' = (12)(12)$$

$x' = 4.5$ cm. (i.e. 4.5 cm to the right of F').

Or the image is $8 + 4.5 = 12.5$ cm to the right of O'. The magnification, m, is given by $m = -12/32 = -3/8$ and so the image is inverted and is $2 \times 3/8 = 3/4$ cm in size.

In summary, we are now able to find corresponding image and object positions for any optical system by:

- i. Determining the product of the refraction and translation matrices between the first vertex O and the last vertex O', the product matrix being designated as $[A]_{OO'}$.
2. Obtaining from $[A]_{OO'}$, the position of the focal points relative to the vertices, (the distances OF and O'F' from equations (23) and (24)) as well as the focal lengths f and f' from equations (28) and (30).
3. Using Newton's equation (33) to compute the positions of the image and object relative to the focal points and equation (35) to compute the magnification.

We shall use the following example to illustrate further these steps and later use the example to illustrate other cardinal points which are discussed in the following sections.

Example 2

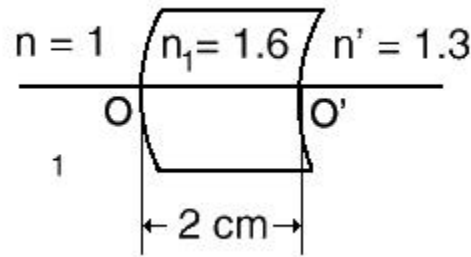


Figure 15.

The thick lens of Fig. 15 has radii of curvature equal to 1.5 cm and OO' is 2 cm. The index of the lens is 1.6, the index to the left of O is 1.00 and the index to the right of O' is 1.30. An object is placed 10.467 cm to the left of the vertex O. Where is the image and what is the magnification?

Solution:

i.

$$[A]_{OO'} = \begin{bmatrix} 1 & 0 \\ 0.4 & 1.6 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ .125 & .8125 \end{bmatrix},$$

or

$$[A]_{OO'} = \begin{bmatrix} 1.25 & -1.625 \\ 0.3 & 0.65 \end{bmatrix}.$$

2. $f = 1/.3 = 3.33$ cm (from equation 28).
 $f' = 1.3 \times f = 4.333$ cm (from equation 30).
 $O'F' = 0.65/0.3 = 2.167$ cm (from equation 23).
 $OF = 1.25/0.3 = 4.167$ cm (from equation 24).
 (See Fig. 16. Also illustrated in the Figure are the principal points, H, H', which will be discussed in the next section)
3. The object is placed 10.467 cm to the left of O so that $x = 10.467 - 4.167$, or $x = 6.3$ cm.
 $x' = ff'/x = (4.333)x(3.333)/6.3 = 2.293$ cm or the image is at a distance $2.293 + 2.167 = 4.460$ cm to the right of O'.
 $m = -f/x = -3.333/6.3 = -0.529$ (from equation (35))
 or
 $m = -x'/f' = -2.293/4.333 = -0.529$

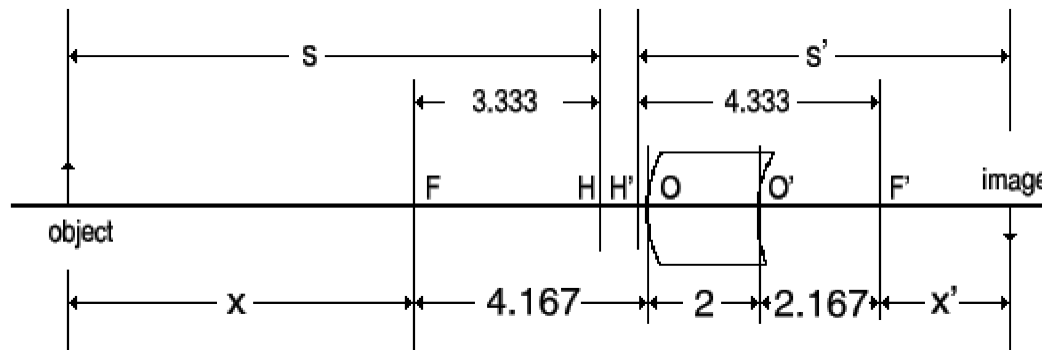


Figure 16.

Chapter 5

Image Formation

The Cardinal Points

It is often convenient to discuss image formation in terms of other positions than the focal points. These other points are the positive and negative principal points and the positive and negative nodal points. These, together with the focal points, are all defined as the cardinal points of a lens system and the planes perpendicular to the optic axis through these points are the cardinal planes. From the basic definitions of principal points and nodal points, it is a simple matter to find their positions using the matrices already described.

The positions of the principal points and nodal points are most readily specified in terms of their positions relative to the focal points F and F'. Let us assume that a cardinal point Z, say, is a distance z from F and the cardinal point Z' is a distance z' from F'. Then the ray y, α , at the plane through Z goes through the plane at Z' with coordinates y', α' . This can be expressed as

$$\begin{bmatrix} y \\ \alpha \end{bmatrix}_Z = \begin{bmatrix} 1 & -z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -f' \\ 1/f & 0 \end{bmatrix}_{FF'} \begin{bmatrix} 1 & -z' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}_{Z'}$$

or

$$\begin{bmatrix} y \\ \alpha \end{bmatrix}_P = \begin{bmatrix} -z/f & z'z/f - f' \\ 1/f & -z'/f \end{bmatrix}_{ZZ'} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}_{Z'}. \quad (36)$$

We may now define the principal and nodal points and the definitions will impose conditions on the matrix of (36) which will give the positions z and z' of the points in question relative to F and F'.

Positive Principal Points

The planes through the positive principal points, H and H', are the positive principal planes and are defined as planes of unit positive magnification, that is, planes for which $y' = y$. But from (36) $y = -(z/f)y + (zz'/f - f')\alpha'$. Thus we must have $zz'/f - f' = 0$ and $-z/f = 1$ for the positive principal points H and H'.

Then

$$z = FH = -f, \quad (37)$$

$$z' = FH' = -f', \quad (38)$$

and

$$\begin{bmatrix} y \\ \alpha \end{bmatrix}_H = \begin{bmatrix} 1 & 0 \\ 1/f & f'/f \end{bmatrix}_{HH'} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}_{H'}. \quad (39)$$

The first and second focal lengths are generally **defined** as the distances FH and F 'H ' respectively. We see then from equations (37) and (38) that we are justified in having written equations (28) and (29) where we introduced the symbols f and f' into the matrix connecting the first and second focal planes.

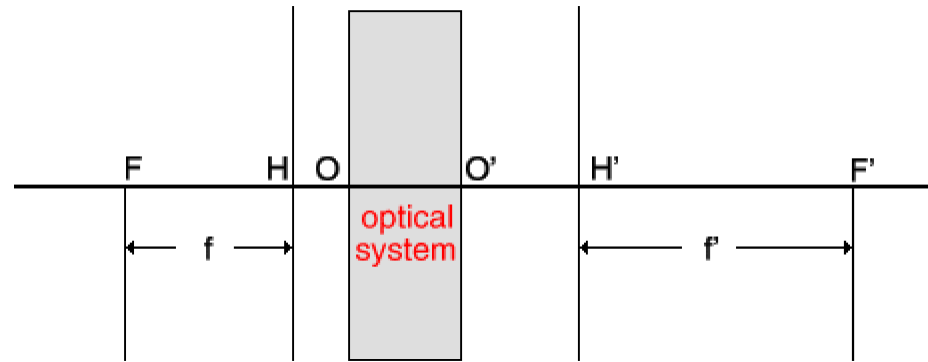


Figure 17.

The principal point, H ', is a distance f' to the left of F ', the principal point H a distance f to the right of F for a positive focal length system as illustrated in Fig. 17. Depending on the system the points may lie on either side of O or O ' and even may be "crossed," that is, H ' may lie to the left of H.

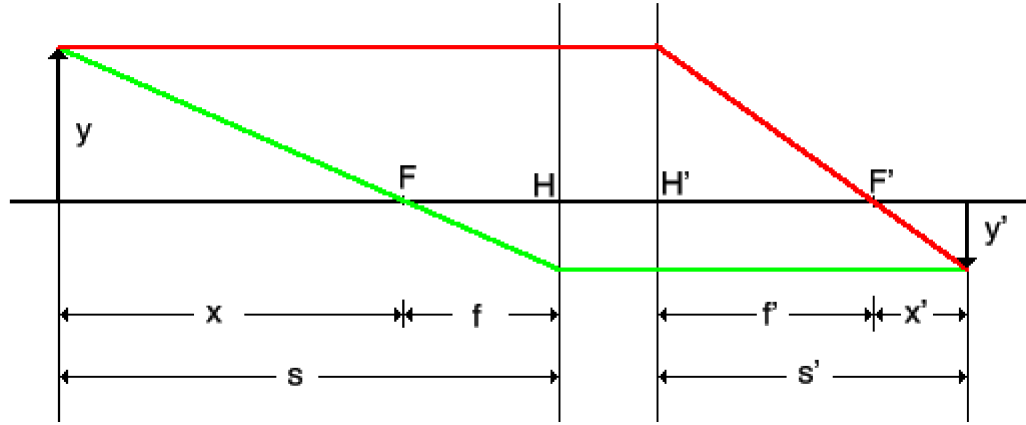


Figure 18.

In Fig. 18 is illustrated a ray diagram showing the formation of an image using the properties of the principal planes and of the focal points. (Equation (34), $y = -x y' / f$, also follows from the diagram.) The object distance as measured from H is s , the image as measured from H' is s' . The ray y', α' at the image is related to the ray y, α , from the object by

$$\begin{bmatrix} y \\ \alpha \end{bmatrix}_s = \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/f & f'/f \end{bmatrix}_{HH'} \begin{bmatrix} 1 & -s' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}_{s'}$$

or

$$\begin{bmatrix} y \\ \alpha \end{bmatrix}_s = \begin{bmatrix} 1 - s/f & -s' + ss'/f - sf'/-f \\ 1/f & (-s' + f')/f \end{bmatrix} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}_{s'}. \quad (40)$$

As stated before in deriving Newton's equations, the upper right hand element of the matrix must be zero if y' is an image of y , that is, if the rays through the image plane at height y' are to be independent of α' . Thus we have

$$f'/s' + f/s = 1 \quad (41)$$

which makes the upper right hand element zero and gives $y = ((f - s)/f)y'$ or the magnification, $m = y'/y = f/(f - s)$. From equation (41), $(f - s) = -sf'/s'$ and so

$$m = -fs'/f's = -ns'/n's, \quad (42)$$

and since the determinant equals f'/f equation (40) becomes

$$\begin{bmatrix} y \\ \alpha \end{bmatrix}_s = \begin{bmatrix} 1/m & 0 \\ 1/f & mf'/f \end{bmatrix} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}_{s'} \quad (43)$$

Example 3

Use (41) and (42) to compute the image positions of the object in [example 2](#).

Solution: From the results of example 2, ([see Fig. 16](#)) H is 3.333 cm to the right of F or (4.167 - 3.333) = .834 cm to the left of O. H' is 4.333 cm to the left of F' or (4.333-2.167) = 2.167 cm to the left of O'. The object distance $s = 10.467 - .834 = 9.633$ and

$$4.333/s' + 3.333/9.633 = 1.$$

so $s' = 6.627$ cm.

Measured from O' the image is $6.627 - 2.167 = 4.46$ cm (to the right of O') in agreement with Newton's equations of the previous example. Also $m = -(1.0/1.30) \times (6.627/9.633) = -0.529$ as before.

If the medium has the same index on either side of the optical system, $f = f'$ and these equations take the very familiar forms,

$$1/s' + 1/s = 1/f, \quad (44)$$

and

$$m = -s'/s. \quad (45)$$

For a thin lens in air, the situation is even simpler as the image and object distances s' and s can be measured from the same point, the centre of the lens, since for a thin lens the principal planes are approximately coincident at the lens centre. This and other special cases will be discussed in the next chapter.

Negative Principal Points

The planes perpendicular to the optic axis through the negative principal points, H(-) and H'(-), are the negative principal planes and are defined as planes of unit negative magnification, that is, $y' = -y$. It is left to the reader to show that

$$z' = F'H'(-) = f', \quad (46)$$

and

$$zFH(-) = f, \quad (47)$$

and

$$\begin{bmatrix} y \\ \alpha \end{bmatrix}_{H(-)} = \begin{bmatrix} -1 & 0 \\ 1/f & -f'/f \end{bmatrix}_{H(-)H'(-)} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}_{H'(-)}. \quad (48)$$

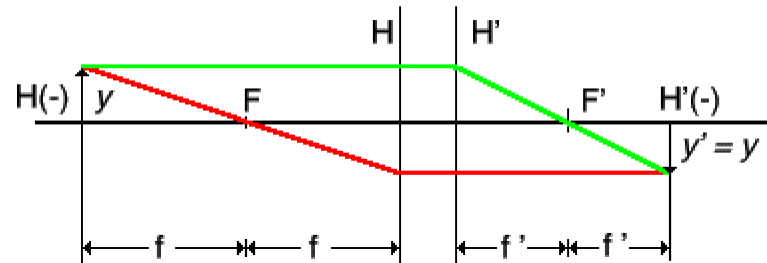


Figure 19.

Thus in the positive system illustrated in Fig. 19, an object placed at the distance f to the left of F will give rise to an inverted image, the same size as the object, at a distance f' to the right of F' .

It is left as a further exercise to the reader to show that the equation for image formation may be expressed in terms of the object distance, u from $H(-)$ and the image distance, u' , from $H'(-)$ as

$$f/u + f'/u' = 1 \quad (49)$$

and

$$m = fu'/f'u. \quad (50)$$

Chapter 6

Practical Optical Systems

The methods developed in the preceding chapters may now be applied to a number of common optical systems. In each case we shall compute the matrix $[A]_{OO'}$, from which all the cardinal points are obtainable. We shall mainly use equations (23), (24), (28), and (30) which give the positions of the focal points and the focal lengths. Then Newton's equations (33) and (35) may be used for image formation, or alternatively equations (41) and (42) which make use of the principal points given by equations (37) and (38).

The Single Refracting Surface

For a single spherical surface separating two media of indices n and n' , the matrix $[A]_{OO'}$ is simply the refraction matrix, namely,

$$\begin{bmatrix} 1 & 0 \\ (n' - n)/nR & n'/n \end{bmatrix}.$$

We can immediately recognize

$$f = nR/(n' - n), \quad (61)$$

and

$$f' = (n'/n)f = n'R/(n' - n). \quad (62)$$

The distances $OF = nR/(n' - n) = f$ and $OF' = (n'/n)nR/(n' - n) = n'R/(n' - n) = f'$. Since (38) gives $F'H' = -f'$ and (37) gives $FH = -f$, the principal points H and H' are here coincident at the vertex O of the refracting surface.

Example 1

As an example, consider [exercise 1 of chapter 3](#) in which an object is placed 24 cm to the left of a vertex O of a convex spherical surface separating media of index 1.00 and 1.50. The radius of curvature of the surface is 6 cm (see Fig. 5). Solution:

$$f' = (1.5)6/(.5) = 18 \text{ cm}$$

$$f = (1/1.5)(18) = 12 \text{ cm}$$

$OF = 12 \text{ cm}$ $OF' = 18 \text{ cm}$

$$x = 24 - 12 = 12 \text{ cm}$$

$$x' = (18)(12)/12 = 18 \text{ cm.}$$

The image distance (from O to the image point) is $18 + 18 = 36$ cm. In this example the image and object positions are in fact the negative principal points since $m = -x'/f = -18/18 = -1$. The nodal point N is a distance f' to the right of F and $ON = f' - f = n'R/(n' - n) - R/(n' - n) = R = 6$ cm. Of course N' is coincident with N, as H' is coincident with H. This example is illustrated in Fig. 23.

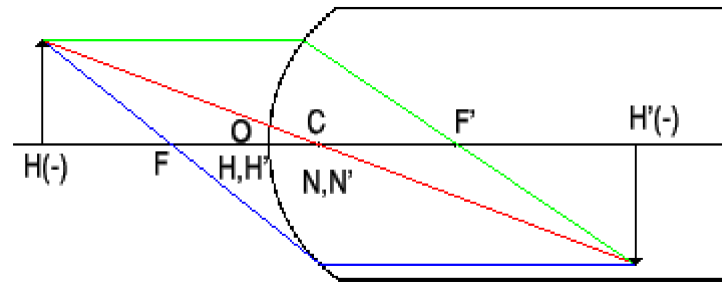


Figure 23.

Example 2

We can do [example 1 in chapter 5](#) by repeated application of the above equations for a single refracting surface. We had a thick biconvex lens, index 1.5 in air, 10 cm thick, with radii of 10 cm. An object was placed 40 cm in front of the first vertex O.

For the first refracting surface, $f' = (1.5)(10)/.5 = 30$ cm, $f = (1/1.5)(30) = 20$ cm, and $30/s' + 20/40 = 1$ gives $s' = 60$ cm. This gives an image 50 cm to the right of the second vertex O' so that the image distance for refraction at the second surface is $s = -50$ cm. For the second surface, $f' = (1)(-10)/(-.5) = 20$ cm, $f = (1.5) 20 = 30$ cm, and $20/s' + 30/(-50) = 1$ gives $s' = 12.5$ cm in agreement with our previous result. And the magnification, $m = -(60/40)(-12.5)/(-50) = -3/8$ as before.

Exercise 1

Find the first and second focal points of the lens of [example 2 in chapter 5](#) by repeated application of the equation for image formation by a single refracting surface. We might note that the plane surface is simply a special case in which $R = \infty$ and the refraction matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & n'/n \end{bmatrix}.$$

The focal lengths are equal to ∞ . It is easy to show that

$$n'/s' + n/s = 0 \quad (63)$$

if s is the object distance measured from O and s' is the image distance also measured from O . For example s' represents the "apparent depth" of an object a distance s under water (index $4/3$) say, such that $s' = (3/4)s$.

Exercise 2

Derive equation (63).

The Thin Lens

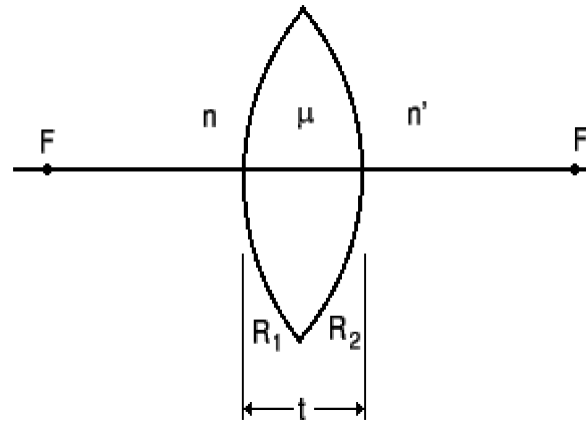


Figure 24.

Possibly the most useful equations of geometrical optics are those for image formation by a thin lens. A thin lens of index μ , is shown in Fig. 24 with media of indices n and n' to the left and right of O and O' respectively. The thickness, t , of the lens is assumed to be small compared to the focal lengths of the lens such that we can set $t = 0$ in the translation matrix from O to O' . Thus

$$[A]_{OO'} = \begin{bmatrix} 1 & 0 \\ (\mu - n)/nR_1 & \mu/n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (n' - \mu)/\mu R_2 & n'/\mu \end{bmatrix}.$$

or

$$[A]_{OO'} = \begin{bmatrix} 1 & 0 \\ (\mu - n)/nR_1 + (n' - \mu)/nR_2 & n'/n \end{bmatrix}_{OO'},$$

(assuming O and O' approximately coincident).

So,

$$1/f = (n' - \mu)/nR_2 - (n - \mu)/nR_1$$

or

$$n/f = n'/f' = (n' - \mu)/R_2 - (n - \mu)/R_1. \quad (64)$$

Equation (64) reduces to the well-known lens-makers equation when $n' = n = 1.00$, namely,

$$1/f = 1/f' = (\mu - 1)(1/R_1 - 1/R_2). \quad (65)$$

Exercise 3

Show that the points H and H' coincide at the assumed approximate coincidence of O and O' which we take to be the lens centre of the thin lens.

Exercise 4

Find the positions of the nodal points of the thin lens.

Newton's equations apply here using (64) for f and f' . It is also convenient to use (41) since s and s' are measured from the centre of the thin lens, that is, $f'/s' + f/s = 1$ or $n'/s' + n/s = n/f = n'/f'$ which for a thin lens in air becomes $1/s + 1/s' = 1/f = 1/f' = (\mu - 1)(1/R_1 - 1/R_2)$. Also $m = -ns'/n's (= -s'/s \text{ if } n' = n)$

The matrix representing a thin lens has the particularly simple form

$$[A]_{OO'} = \begin{bmatrix} 1 & 0 \\ 1/f & f'/f \end{bmatrix}. \quad (66)$$

Question

A zero in the upper right hand corner of the matrix means that the matrix relates an image plane and an object plane. What does the zero mean in Equation (66)?

Combination of Two Thin Lenses

The combination of two thin lenses in air forms the basic arrangement for the most common optical systems such as the microscope and astronomical refracting telescope. Also, most oculars are combinations of two lenses as corrections for some of the defects of images can be reduced in this way. The defects are classed in terms of five monochromatic aberrations arising as correction terms to the first order theory, plus two so-called chromatic aberrations arising from the dependence of the index of refraction on frequency.

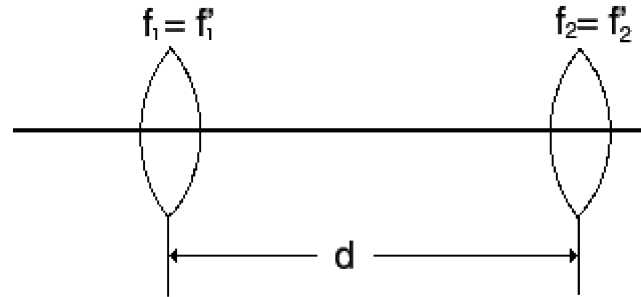


Figure 25.

The combination of two thin lenses in air, a distance d apart, is shown in Fig. 25. We may take the vertices of the system as the lens centres O_1 and O_2 . The matrix is then, using (66) with $f_1 = f_1'$, and $f_2 = f_2'$

$$\begin{aligned} A_{O_1 O_2} &= \begin{bmatrix} 1 & 0 \\ 1/f_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/f_2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 - d/f_2 & -d \\ 1/f_2 + (f_2 - d)/f_1 f_2 & 1 - d/f_1 \end{bmatrix}_{O_1 O_2}. \end{aligned}$$

Therefore if $f (= f')$ is the focal length of the combination, equation(28) gives,

$$1/f = 1/f_1 + 1/f_2 - d/f_1 f_2. \quad (67)$$

The focal points are F and F' such that

$$O_1 F = f(1 - d/f_2) \quad (68)$$

and

$$O_2 F = f(1 - d/f_1) \quad (69)$$

using (23) and (24).

One simple special case of interest is that in which two thin lenses are cemented together to form a doublet. In this case $d = 0$ and $1/f = 1/f_1 + 1/f_2$. Such a doublet is usually constructed to reduce longitudinal chromatic aberration. The lenses are made of different glasses which have different dispersive powers (i.e. the indices of the glasses depend differently on the frequency of the light) and so the combination may be designed so that the focal length f will have a minimum dependence on the frequency of the light (Jenkins and White, p. 157). It is also possible to reduce the dependence of the focal length, f , on the frequency when using two lenses of the same index of refraction by choosing a separation $d = (f_1 + f_2)/2$. This may be readily shown using (67) and (65) and solving for a minimum dependence of f on μ , the index of the glass of the lenses (Jenkins and White, p. 163).

Chapter 6

Practical Optical Systems

The Telescope

The telescope in its most basic form is a special case of two thin lenses in air. In this form it is a simple astronomical refracting telescope. Basically, a telescope is used to increase the angle that a distant object subtends at the eye. If the eye is relaxed for distant viewing, the telescope simply produces an angular magnification in which an incident (approximately) parallel beam from a point in a distant source, making an angle α say with respect to the optic axis, emerges as a parallel beam making a larger angle α' with respect to the axis. When the object is viewed with the instrument, the image formed on the retina of the eye will be larger by an amount proportional to α'/α , the angular magnification, M . This telescopic system is characterized by a zero in the lower left hand corner of the matrix, $[A]_{O_1, O_2}$, which we obtain if $d = f_1 + f_2$. Then, from $\{A\}_{O_1, O_2}$ above

$$\begin{bmatrix} y \\ \alpha \end{bmatrix} = \begin{bmatrix} -f_1/f_2 & -(f_1 + f_2) \\ 0 & -f_2/f_1 \end{bmatrix}_{O_1, O_2} \begin{bmatrix} y' \\ \alpha' \end{bmatrix}_{O_2} \quad (70)$$

or $M = \alpha'/\alpha = -f_1/f_2$. Thus the angular magnification is the negative ratio of the focal length of the first lens, the objective lens, to the focal lengths of the second lens, the eyepiece lens. As illustrated in Fig. 26 the image as viewed through the telescope is inverted, and the image formed by the objective lens is in the second focal plane of that lens which is also the first focal plane of the eyepiece lens.

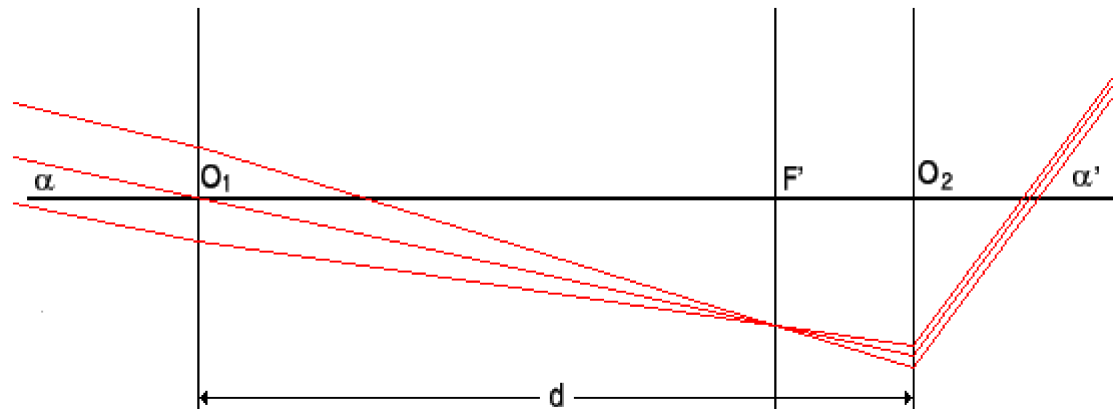


Figure 26.

For a terrestrial telescope one may insert an erector lens in between the objective and eyepiece lenses such that the image formed by the objective acts as an object for the erector lens which in turn forms an inverted image at the first focal point of the eyepiece lens. The matrix of

this system of three lenses will have a zero in the lower left hand corner and the angular magnification is positive.

Exercise 5

Find the matrix of the system of the three lenses of Fig. 27 and show that they form a telescopic system with positive angular magnification. Solve for the case in which $f_1 = 20$ cm, $f_i = 2$ cm and $f_2 = 5$ cm. (Answer $M = +4$)

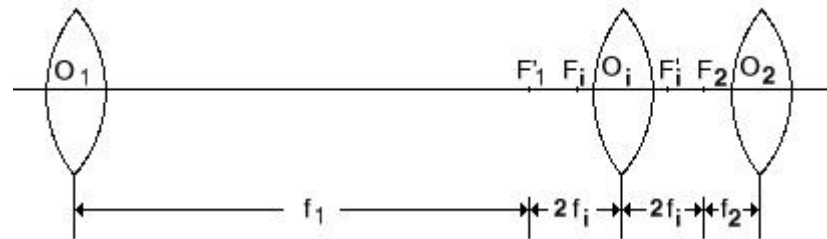


Figure 27.

Exercise 6

A Galilean telescope has an objective lens with $f = 20$ cm and the eyepiece lens with $f = -5$ cm. The lenses are separated by 15 cm. Calculate the matrix for this system and show that it is a telescopic system with $M = +4$.

The Ocular

In most telescopes it is common practice to have a system of at least two lenses called an ocular to perform the function of the eyepiece lens. The first lens or field lens improves the field of view and in combination with the second lens or eye lens can be chosen to reduce some of the aberrations inherent in a single lens. The most common oculars are the Ramsden and Huygen types.

Exercise 7

A Ramsden eyepiece consists of two equal focal length lenses, f_1 . If the separation is equal to f_1 , chromatic aberration is minimized. In practice the lenses are usually moved closer together (Jenkins and White, p. 182).

Calculate the position of the focal points and the focal lengths of the ocular in the case in which $d = (3/4) f_1$.

Magnifiers

The ocular discussed above as well as just a single thin lens can be used as a magnifying glass. When an object is examined an image is formed at the retina in the eye and this retinal image may be increased in size using a magnifier. For continued viewing, it is usual to assume that an object 25 cm in front of the eye can be comfortably viewed. For this case then, an object of height y subtends an angle θ_o at the eye

with $\theta_o = y/25$, as in Fig. 28. Now if the object is placed at the focal point of an ocular, the emergent rays from the ocular may be focused by a relaxed eye as also shown in Fig. 28. The rays from y now make an angle of $\theta_i = y/f$ where f is the focal length of the ocular. Thus the change in the size of the retinal image is given approximately by $\theta_i/\theta_o = (y/f)/(y/25) = 25/f$.

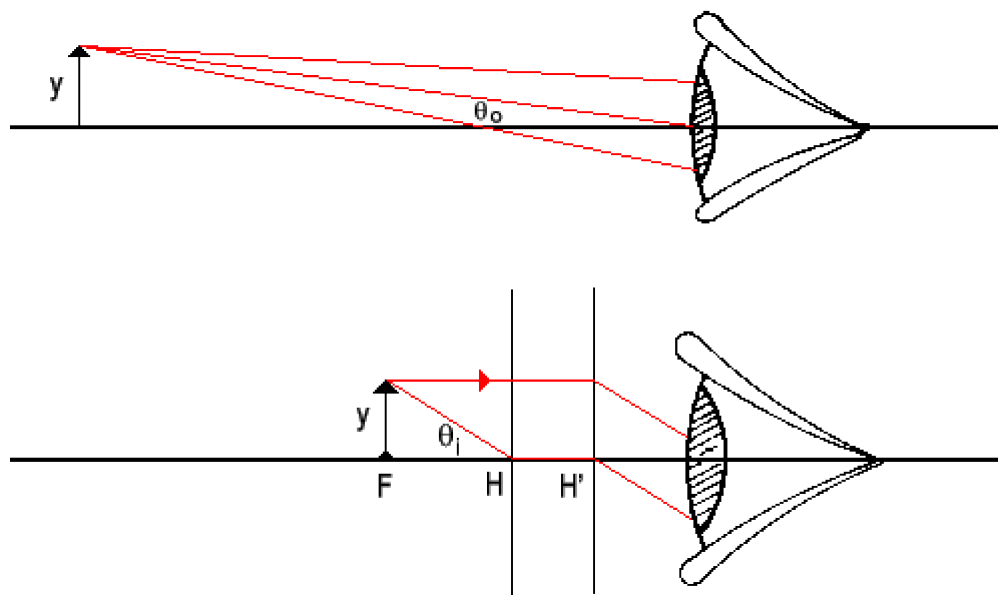


Figure 28.

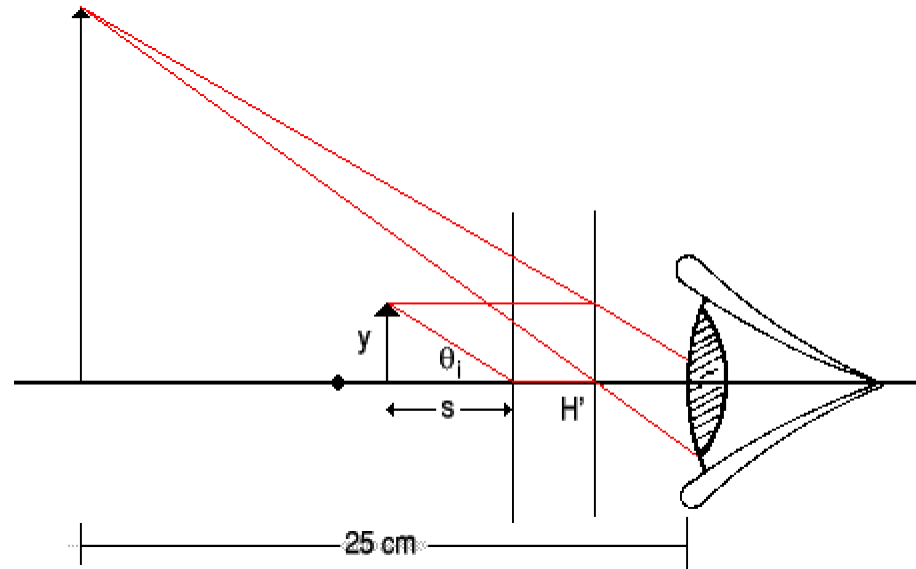


Figure 29.

The ocular may also be used as in Fig. 29 with the object just inside the focal point of the ocular for an eye accommodated to the 25 cm viewing distance. In this case the magnifying power (which is usually quoted on the eyepiece) is $(1 + 25/f)$. This is an approximate formula which assumes that the image distance from the second principal point is 25 cm and then

$$M = \frac{\theta_i}{\theta_o} = \frac{y/s}{y/25} = \frac{25}{s} = 25 \left(\frac{1}{f} - \frac{1}{s'} \right) = \frac{25}{f} + 1 \text{ (since } s' = 25 \text{ cm).}$$

Exercise 8

Calculate the magnifying power for the Ramsden eyepiece of the above exercise 7 for both a relaxed eye and an accommodated eye if $f_1 = 2$ cm.

Chapter 7

Spherical Mirrors

Convex and concave spherical reflecting surfaces have image-forming qualities which can be discovered from the laws of reflection stated in chapter 1. In what follows we shall apply the matrix method to the case of reflection, again limiting consideration to paraxial rays.

The difficulty that often arises in discussing image formation by mirrors is that of applying the sign convention used for refraction. In the case of refraction the rays continue in the same (positive x) direction after refraction but in reflection the rays are of course reversed. To avoid the difficulties, one may define a different convention for reflection or specify the positive image direction in terms of the on-going emergent rays. Or, in terms of the mathematical formulation, one can regard reflection as a special case of refraction in which Snell's law, $n \theta = n' \theta'$, (for small angles), becomes the law of reflection, $\theta = -\theta'$ for $n = -n'$.

In the matrix formulation it is perhaps easier to maintain all our previous sign conventions for refraction and treat the reflected rays as having a slope angle equal to $-\alpha'$. This in effect allows us to treat the rays after reflection as if they were proceeding on in the same direction as the incoming rays. Of course the final calculated quantities such as a positive image distance or a positive focal distance, for example, will actually lie to the left of the reference point in constructing a ray diagram. The actual path of a ray after reflection from a convex mirror is shown in Fig. 30 and the corresponding diagram for deriving the reflection matrix is shown in Fig. 31.

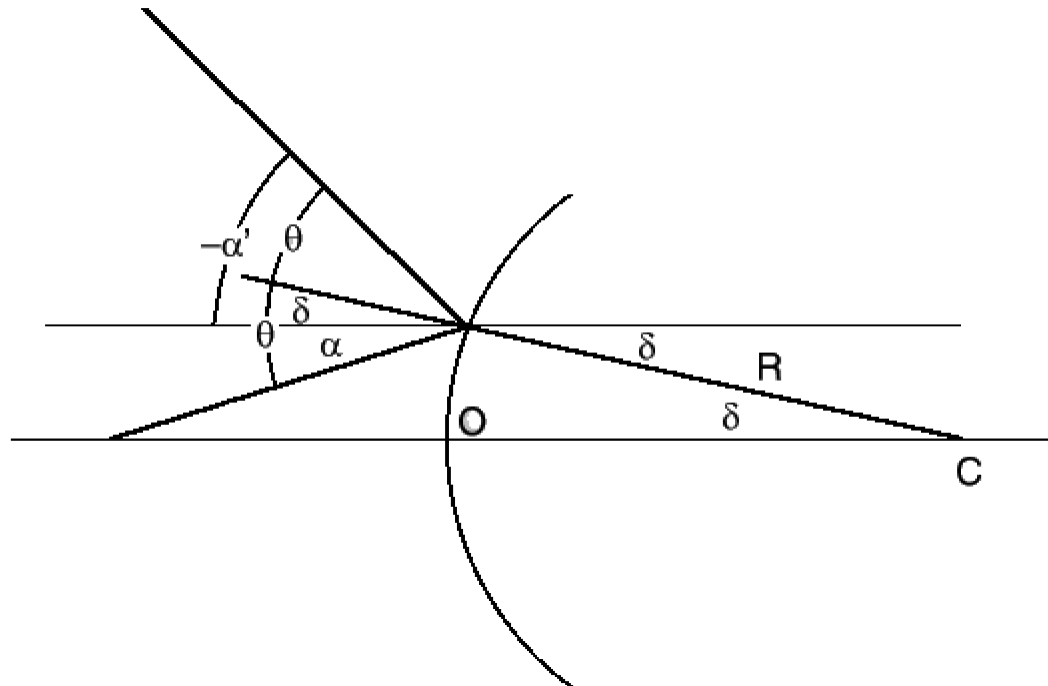


Figure 30.

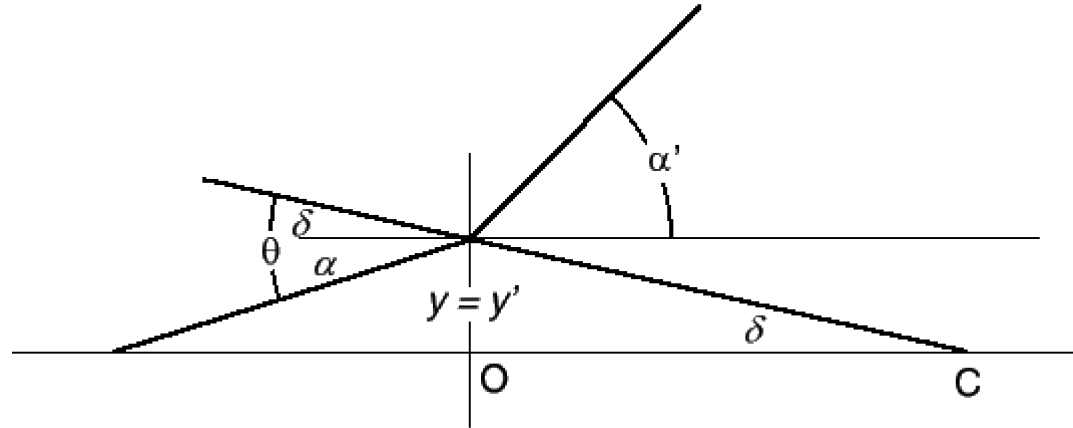


Figure 31.

The transformation equations for reflection are easily derived from Fig. 31. From the figure,

$$\theta = \delta + \alpha$$

and

$$\alpha' = \theta + \delta.$$

Therefore

$$\alpha' = 2\delta + \alpha = 2y/R + \alpha$$

since

$$\delta = y/R = y'/R.$$

The transformation equations are then,

$$y = y' + 0\alpha'$$

$$\alpha = -2/Ry' + \alpha'$$

or

$$\begin{bmatrix} y \\ \alpha \end{bmatrix}_o = \begin{bmatrix} 1 & 0 \\ -2/R & 1 \end{bmatrix} \begin{bmatrix} y' \\ \alpha \end{bmatrix}$$

where

$$\begin{bmatrix} 1 & 0 \\ -2/R & 1 \end{bmatrix}$$

is the reflection matrix.

The Single Reflecting Surface

For a single reflecting surface the $[A]_{OO}$ matrix refers simply to the vertex of the reflecting surface and from (28) with $[A]_{OO}$ equal to the reflection matrix,

$$f = -R/2 \quad (71)$$

and from (30) $f' = f$ as the matrix determinant is unity. Of course Newton's equations for image formation apply as always and may be written here as

$$xx' = R^2/4$$

and

$$m = R/2x = 2x'/R.$$

Alternatively,

$$f'/s' + f/s = 1$$

becomes

$$1/s' + 1/s = -2/R$$

and

$$m = -s'/s.$$

The distances x and x' , s and s' , and R follow the same convention as before for refraction. The final answer however must then be "reflected" back to its true position. Let us illustrate this with the following example:

Example 1

A concave mirror has a radius of 12 cm and an object 2 cm high is placed 24 cm to the left of the vertex O of the mirror. Where is the image formed and how large is it?

Solution: (1) $f = -R/2 = -(-12)/2 = 6$ cm.

OF is 6 cm to the left of O.

OF' is 6 cm to the right of O.

Since $x = 18$ cm, $x'(18) = 36$, $x' = 2$ cm to the right of F' or 8 cm to the right of O (I' (calc) in Fig. 32).

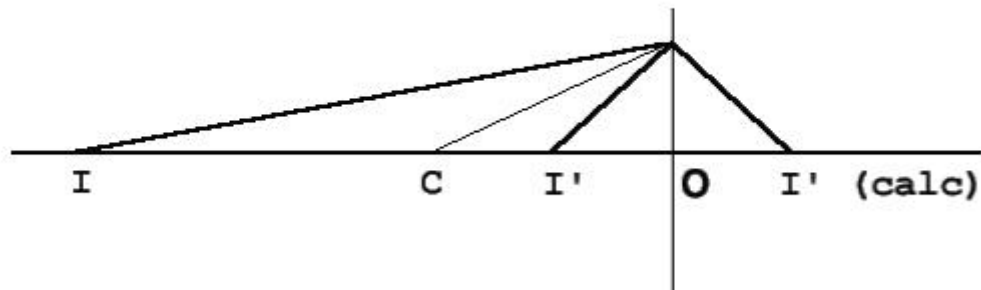


Figure 32.

Now the actual rays pass through the point 8 cm to the left of O, the point I' in Fig. 32. The magnification $m = -f/x = -6/18$ and therefore the image is inverted and $2/3$ cm in size.

(2) We may also use

$$1/s' + 1/s = -2/R$$

which gives

$$1/s' = (-2/-12) - 1/24 = 1/6 - 1/24 = 1/8.$$

Then $s' = 8$ cm to the right of the mirror plane at O (I' (calc) in Fig. 32) or the real image is 8 cm to the left of O, (I' in Fig. 32).

Chapter 7

Spherical Mirrors

A Thick Mirror

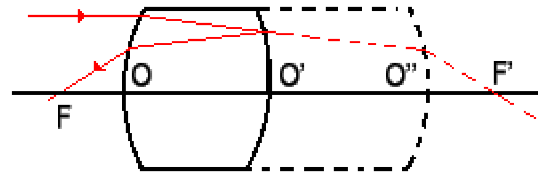


Figure 33.

A thick lens with the second surface covered with a reflecting coating as in Fig. 33 can be treated by the matrix method. The rays on reflection are considered by the reflection transformation above to be proceeding in the positive x direction so that the optical system to the left of the reflecting surface at O' must be "reflected" at O' as well, as shown in Fig. 33. The final distances in the problem are referred to O'' , that is, a positive distance to the right of O'' is actually the same distance to the left of O .

Example 2

A biconvex lens 2 cm thick with a radius of 6 cm is silvered on the second surface. The index of refraction of the lens material is 1.5. Where are the focal planes of this thick mirror?

Solution:

$$\begin{aligned} [\mathbf{A}]_{oo''} &= \begin{bmatrix} 1 & 0 \\ 1/12 & 3/2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/18 & 2/3 \end{bmatrix} \\ &= \begin{bmatrix} 5/27 & -16/9 \\ 44/81 & 5/27 \end{bmatrix}. \end{aligned}$$

Thus $f = 81/44 \text{ cm} = f'$.

$OF = (81/44) (5/27) = 15/44 \text{ cm}$ and

O''F' = 15/44 cm using equations (23) and (24).

The ray diagram is shown in Fig. 33.

Example 3.

As a further example of the use of the focal planes, suppose an object 2 cm high is placed 15/22 cm in front of the vertex O. Where is the image and how large is it?

Solution:

Since $x = 30/44 - 15/44 = 15/44$

$$(15/44)x' = (81/44)(81/44)$$

or $x' = 9.94$ cm, or $9.94 + 15/44 = 10.28$ cm to the right of O'', which is actually 10.28 cm to the left of O. Since $m = - (81/44)/(15/44) = -5.4$, the image is 10.8 cm in size and inverted (or $m = -x'/f' = -9.94/1.835 = -5.4$).

Exercise

Solve the above example to find the focal positions using repeatedly the equations for reflection and refraction at a single surface.

Astronomical Reflecting Telescope

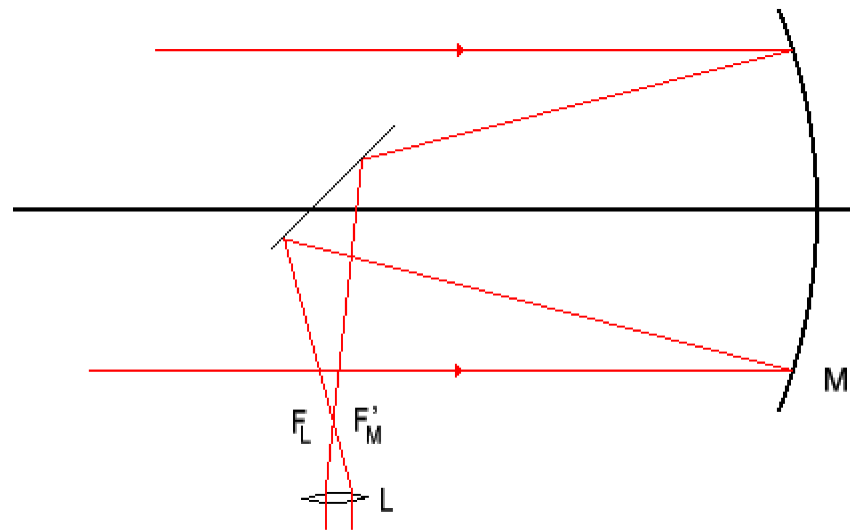


Figure 34.

One simple form of the astronomical reflecting telescope is shown in Fig. 34. A parallel beam of light reflected from the spherical concave mirror is brought to the second focal point, F'_M , of this mirror after further reflection by a small plane mirror. The focal point F' is also the first focal point F_L of the eyepiece lens. The plane mirror is sufficiently small so that it interrupts only a small amount of the light falling on the concave mirror. In another common arrangement, the spherical mirror has a relatively small hole in its centre and a small plane (or convex) mirror redirects the light back to a focus at the centre of the large mirror where the focal image is examined with the eyepiece.

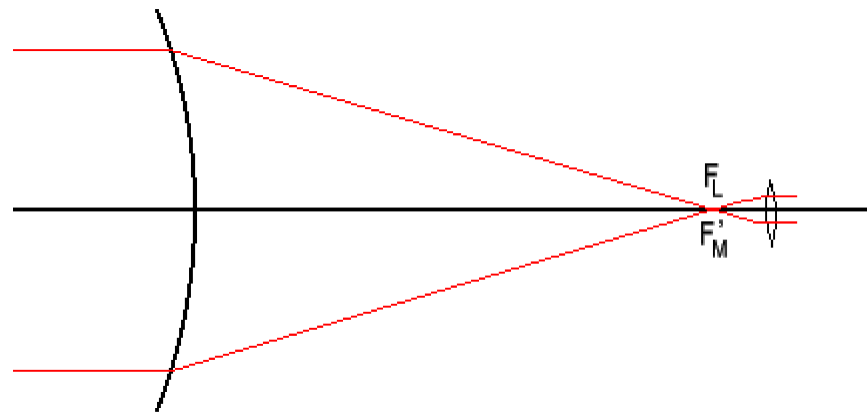


Figure 35.

The matrix method may be used to discuss this telescope as indicated in Fig. 35. Then, if $|R|$ is the absolute value of the radius of curvature of

the mirror and $f_L = f'_L$ equals the focal length of the lens

$$[\mathbf{A}]_{oo'} = \begin{bmatrix} 1 & 0 \\ -2/R & 1 \end{bmatrix} \begin{bmatrix} 1 & -(|R|/2 + f_L) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/f_L & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} R/2f_L & -(|R|/2 + f_L) \\ 0 & 2f_L/R \end{bmatrix}.$$

The telescopic system is again indicated by the zero in the lower left hand corner of the matrix and the angular magnification, $M = \alpha'/\alpha = R/2f_L$.

In practice, the large reflecting telescopes are usually used simply as cameras with a photographic plate placed in the focal plane of the large reflector. The surface of the reflector is made parabolic in shape to avoid spherical aberrations. Of course a reflector is free from chromatic aberrations as the laws of reflection are the same for all frequencies of light. The mirror may be made large in diameter in order to attain a high angular resolving power and to increase the light flux per unit area (the illuminance) at the photographic plate.