Component Redundancy *vs* System Redundancy in the Hazard Rate Ordering

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Key Words — **Component redundancy, system redundancy, coherent system, stochastic ordering, hazard rate ordering.**

Summary & Conclusions — **Design engineers are well aware of the stochastic result which says that (under the appropriate assumptions) redundancy at the component level is superior to redundancy at the system level. Given the importance of the hazard rate in reliability and life testing, we investigate to what extent this principle holds for the stronger stochastic ordering, *viz*, hazard rate ordering. Surprisingly, this does not hold for even *series* systems if the spares do not match the original components in distribution. It is true for *series* systems however for matching spares, and we conjecture that this is the case in general for k-out-of-n:G systems. We also investigate this principle for cold-standby redundancy (as opposed to active or *parallel* redundancy).**

1. INTRODUCTION

Notation

n	number of components in s-coherent system
T_i	lifetime of component i in s-coherent system, $i=1,\ldots,n$; r.v.
$\boldsymbol{T}$	$(T_1,\ldots,T_n)$: s-independent lifetimes of n components in system; r.v.
$\boldsymbol{T}^*$	$(T_1^*,\ldots,T_n^*)$: s-independent lifetimes of an s-independent set of n spares for system; r.v.
τ	$\tau(\boldsymbol{T})$: lifetime of s-coherent system with component lifetimes $\boldsymbol{T}$; r.v.
$F_i(t)$, $\bar{F}_i(t)$	Cdf$\{T_i\}$, Sf$\{T_i\}$
$r_\tau(t)$	hazard rate for τ
$h(\bar{F}_1(t),\ldots,\bar{F}_n(t))$	system reliability at time t with components $1,\ldots,n$
$h_{\mathcal{Z}}(\bar{F}(t))$	reliability of a $\mathcal{Z}$ system at time t, with $\bar{F}_i(t)\equiv\bar{F}(t)$; $\mathcal{Z}$ is a specified k-out-of-n system
$\stackrel{d}{=}$	implies: equal in distribution
$X\vee Y$	$\max(X,Y)$
$\boldsymbol{T}\vee\boldsymbol{T}^*$	$(T_1\vee T_1^*,\ldots,T_n\vee T_n^*)$
$\stackrel{st}{\geq}$	implies: 'larger than or equal to, in stochastic ordering' [2]
$\stackrel{hr}{\geq}$	implies: 'larger than or equal to, in hazard rate ordering'
$'$	implies: d/dt.

Other, standard notation is given in "Information for Readers & Authors" at the rear of each issue.

The terms, *series* & *parallel* are used in their logic-diagram sense, irrespective of the schematic-diagram or physical-layout. Thus, for n components, a *series* system is 1-out-of-n:F, and a *parallel* system is 1-out-of-n:G. Hot, active, and *parallel* redundancy are different names for the same thing.

Assumptions

1a. The states of all components are mutually s-independent.

1b. The states of all systems, in system-level redundancy, are mutually s-independent.

2. The reliability of each & every component is not affected by the amount or kind of redundancy.

3. The systems are always s-coherent, regardless of the type of redundancy. ◄

A principle well known to design engineers is that (under appropriate assumptions) redundancy at the component level is more effective than redundancy at the system level [1].

A more formal statement is: The reliability of 'a system of mutually s-independent components which has been enhanced by 1 *parallel* redundant component on each of its components' exceeds at any point in time t the reliability achieved by creating, '1 *parallel* redundant duplicate system on the original system'. ◄

The formal statement is: Let n components form a s-coherent system with s-coherent life function τ, then

$$\Pr\{\tau(T_1\vee T_1^*,T_2\vee T_2^*,\ldots,T_n\vee T_n^*) > t\}$$
$$\geq \Pr\{\tau(T_1,T_2,\ldots,T_n)\vee\tau(T_1^*, T_2^*,\ldots,T_n^*) > t\},$$
$$\text{for all } t \geq 0.$$

This inequality means:

$$\tau(\boldsymbol{T}\vee\boldsymbol{T}^*) \stackrel{st}{\geq} \tau(\boldsymbol{T}) \vee\tau(\boldsymbol{T}^*), \tag{1}$$

$\tau(\boldsymbol{T}\vee\boldsymbol{T}^*)$ is the lifetime of the system after spare i is connected in *parallel* to component i for all $i = 1,\ldots,n$ (active redundancy at the component level).

$\tau(\boldsymbol{T})\vee\tau(\boldsymbol{T}^*)$ is the lifetime of the system formed by using the spare components to replicate the original system and connecting it in *parallel* to the original one (active redundancy at the system level).

Hence inequality (1) describes, in terms of stochastic ordering, precisely what it means that 'redundancy at the component level is better than redundancy at the system level'.

Although '$\stackrel{st}{\geq}$' is a useful way of comparing 2 r.v., there are other senses in which 1 r.v. can be deemed to be greater than another r.v. [2]. Due to the importance of the hazard (failure) rate in reliability, the hazard rate ordering has become more prominent as a method of comparing random lifetimes [3]:

$X \overset{\text{hr}}{\geq} Y$ if $r_X(t) \leq r_Y(t)$ for all t.

For lifetimes X & Y with defined pdf's, there are other useful characterizations of the hazard rate ordering, *eg*,

$X \overset{\text{hr}}{\geq} Y$

$\Leftrightarrow \bar{F}_X(t)/\bar{F}_Y(t)$ is nondecreasing in t

$\Leftrightarrow \Pr\{X-t > x | X > t\} \geq \Pr\{Y-t > x | Y > t\}$

for all t and $x \geq 0$.

This last characterization of hazard rate ordering shows that: $X \overset{\text{hr}}{\geq} Y \Leftrightarrow$ residual lifetime of X given that it has survived t is stochastically greater (in the usual stochastic order) than the residual lifetime of Y given that it has survived time t, for all $t \geq 0$. Hence, $X \overset{\text{st}}{\geq} Y$ makes a statement about the distributions of 2 'new' units with lifetimes X & Y, whereas $X \overset{\text{hr}}{\geq} Y$ makes a statement about X & Y at any age t. Clearly the '$\overset{\text{hr}}{\geq}$' is both a stronger and more informative ordering than '$\overset{\text{st}}{\geq}$' for r.v.

It is reasonable therefore to ask: Is redundancy at the component level better than redundancy at the system level in the hazard rate ordering? More precisely is it true that

$$\tau(\boldsymbol{T} \vee \boldsymbol{T}^*) \overset{\text{hr}}{\geq} \tau(\boldsymbol{T}) \vee \tau(\boldsymbol{T}^*)$$

for any set of component lifetimes $\boldsymbol{T}$, spares $\boldsymbol{T}^*$ and s-coherent structure lifetime τ? Section 2 shows by an example with *series* systems that the answer is in general no. Section 3, however, gives many positive results when the $\boldsymbol{T}^*$ are i.i.d. with $\boldsymbol{T}$.

2. REDUNDANCY WITH NON-IDENTICAL SPARES

Reformulate the problem. Given $\boldsymbol{T}$ & $\boldsymbol{T}^*$, what are the (s-coherent) structures for which $\tau(\boldsymbol{T} \vee \boldsymbol{T}) \overset{\text{hr}}{\geq} \tau(\boldsymbol{T}) \vee \tau(\boldsymbol{T}^*)$? Let τ represent the lifetime of a 1-out-of-n:G system, then trivially this result holds since,

$$\tau(\boldsymbol{T} \vee \boldsymbol{T}^*) = \tau(\boldsymbol{T}) \vee \tau(\boldsymbol{T}^*) = T_1 \vee T_2 \vee \ldots \vee T_n \vee T^*_1 \vee T^*_2 \vee \ldots \vee T^*_n.$$

However example 1 shows that when T^*_i & T_i are not i.i.d. (non matching spare), then even for a 1-out-of-2:F system it is not necessarily true that

$$\tau(\boldsymbol{T} \vee \boldsymbol{T}^*) \overset{\text{hr}}{\geq} \tau(\boldsymbol{T}) \vee \tau(\boldsymbol{T}^*).$$

Example 1

Consider a 1-out-of-2:F system. Let T_1, T_2 be mutually s-independent exponential lifetimes with mean 1, and T^*_1, T^*_2 be s-independent exponential lifetimes (for spares) with mean 0.5. Let τ be the lifetime of a 1-out-of-2:F system; it is not true that

$$\tau(\boldsymbol{T} \vee \boldsymbol{T}^*) \overset{\text{hr}}{\geq} \tau(\boldsymbol{T}) \vee \tau(\boldsymbol{T}^*).$$

Notation

$r_{\text{CR}}(t)$, $r_{\text{SR}}(t)$ hazard rate for 1-out-of-2:F system with [component, system] redundancy.

$r_{\text{CR}}(t) > r_{\text{SR}}(t)$ for large t.

Figure 1 shows a graph comparing these two functions in this instance (smaller hazard rates are better); although initially component-redundancy is better (as measured by hazard rate), eventually system-redundancy is better.

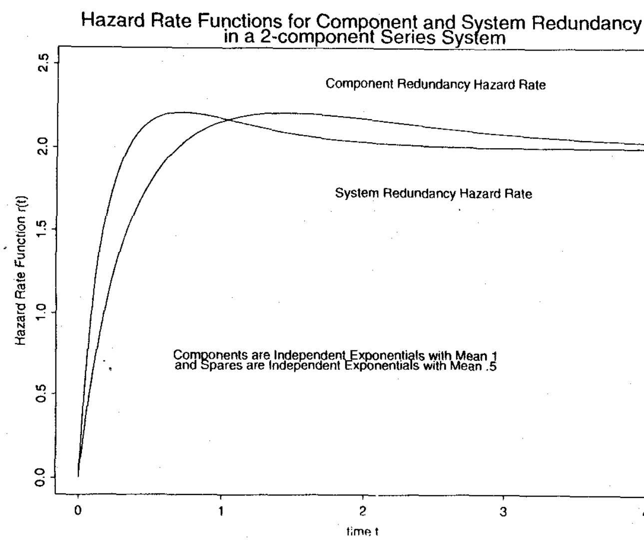

Figure 1. Hazard Rates for Component & System Redundancies in a 1-out-of-2:F System ◄

Example 1 suggests restricting consideration to the situation where spares are matching: $\boldsymbol{T} \overset{\text{d}}{=} \boldsymbol{T}^*$. This is done in section 3.

3. REDUNDANCY WITH MATCHING SPARES

Assumption

4. The spares for the s-coherent system are i.i.d with the original components:

$T_i \overset{\text{d}}{=} T^*_i$ for $i = 1,\ldots,n$.

Theorem 1 shows that, for assumption 4 and for 1-out-of-n:F systems, 'redundancy at the component level' $\overset{\text{hr}}{\geq}$ 'redundancy at the system level'.

Theorem 1

(Use assumptions 1 - 4, and Notation list.)
Let the structure be 1-out-of-n:F. Then,

$$\tau_{1\text{-}n:\mathrm{F}}(\boldsymbol{T}\vee\boldsymbol{T}^*) = \min(T_1\vee T_1^*,\ldots,T_n\vee T_n^*)$$

$$\overset{\mathrm{hr}}{\geq} [\min(T_1,\ldots,T_n)]\vee[\min(T_1^*,\ldots,T_n^*)]$$

$$= \tau_{1\text{-}n:\mathrm{F}}(\boldsymbol{T})\vee\tau_{1\text{-}n:\mathrm{F}}(\boldsymbol{T}^*).$$ ◄

Corollary 1-1 extends theorem 1 to *series-parallel* systems.

Corollary 1-1

(Use assumptions 1 - 4, and Notation list.)
Let $\tau(T_1,T_2,\ldots,T_n) = \min_{1\le j\le m}(\max_{i\in A_j}(T_i))$; where $A_1,\ldots,A_m$ are disjoint, and $\cup_{j=1}^m A_j = \{1,2,\ldots,n\}$, *ie*, the structure is *series-parallel*, Then,

$$\tau(\boldsymbol{T}\vee\boldsymbol{T}^*) \overset{\mathrm{hr}}{\geq} \tau(\boldsymbol{T})\vee\tau(\boldsymbol{T}^*).$$ ◄

Theorem 2 assumes that the $\boldsymbol{T}^*$ 'match' the $\boldsymbol{T}$, and that, in addition, the components are i.i.d.

Theorem 2

(Use assumptions 1 - 4, and Notation list.)
Let: a) all the r.v. be i.i.d; b) n components form a s-coherent system; c) $p\cdot h'(p)/h(p)$ is nonincreasing in p; and d) $h(p) \le p$ for all $p\in[0,1]$, $h(p) \equiv h(p,\ldots,p)$. Then,

$$\tau(\boldsymbol{T}\vee\boldsymbol{T}^*) \overset{\mathrm{hr}}{\geq} \tau(\boldsymbol{T})\vee\tau(\boldsymbol{T}^*).$$ ◄

Remark 1

Let $h(p) = p(\Pi_{j=1}^m h_j(p))$, h_j = reliability of a k_j-out-of-n_j:G system; *ie*, the system is built by connecting in *series* k-out-of-n:G subsystems where at least 1 subsystem consists of 1 component. Then theorem 2 holds since $h(p) \le p$ and $p\cdot h'(p)/h(p)$ is decreasing in p. ◄

For a k-out-of-n:G system with reliability function $h_{k\text{-}n:\mathrm{G}}(p)$, there exists a unique $0 < p^*_{k\text{-}n:\mathrm{G}} < 1$ such that $h_{k\text{-}n:\mathrm{G}}(p) \le p \Leftrightarrow p \le p^*_{k\text{-}n:\mathrm{G}}$ whenever $1 < k < n$ [4]. Hence theorem 2 does not apply to k-out-of-n:G systems when $1 < k < n$. However, the proof indicates that for $t > \bar{F}^{-1}(p^*_{k\text{-}n:\mathrm{G}})$:

$$\Pr\{\tau_{k\text{-}n:\mathrm{G}}(\boldsymbol{T}\vee\boldsymbol{T}^*) > t+x \mid \tau_{k\text{-}n:\mathrm{G}}(\boldsymbol{T}\vee\boldsymbol{T}^*) > t\}$$

$$\geq \Pr\{\tau_{k\text{-}n:\mathrm{G}}(\boldsymbol{T})\vee\tau_{k\text{-}n:\mathrm{G}}(\boldsymbol{T}^*) > t+x \mid \tau_{k\text{-}n:\mathrm{G}}(\boldsymbol{T})\vee\tau_{k\text{-}n:\mathrm{G}}(\boldsymbol{T}^*)$$

$$> t\} \text{ for all } x \ge 0.$$

Hence for a k-out-of-n:G system with i.i.d. components & spares, redundancy at the component level is eventually better than redundancy at the system level in the hazard rate ordering sense.

We conjecture that $\tau_{k\text{-}n:\mathrm{G}}(\boldsymbol{T}\vee\boldsymbol{T}^*) \overset{\mathrm{hr}}{\geq} \tau_{k\text{-}n:\mathrm{G}}(\boldsymbol{T})\vee\tau_{k\text{-}n:\mathrm{G}}(\boldsymbol{T}^*)$ holds in general when components & spares are i.i.d. Theorem 3 proves this result for 2-out-of-n:G systems.

Theorem 3

(Use assumptions 1 - 4, and Notation list.)
Let the lifetimes of n components and n spares be i.i.d. Then,

$$\tau_{2\text{-}n:\mathrm{G}}(\boldsymbol{T}\vee\boldsymbol{T}^*) \overset{\mathrm{hr}}{\geq} \tau_{2\text{-}n:\mathrm{G}}(\boldsymbol{T})\vee\tau_{2\text{-}n:\mathrm{G}}(\boldsymbol{T}^*).$$ ◄

Remark 2

Consider s-coherent systems with 3 i.i.d. components, then the only system which has not been covered by our results so far is the *parallel-series* system defined by,

$$\tau(T_1,T_2,T_3) = [\min(T_1,T_2)]\vee T_3.$$

The reliability of this system is:

$$h(p) = p+p^2-p^3.$$

The statement: 'Redundancy at the component level' $\overset{\mathrm{hr}}{\geq}$ 'redundancy at the system level'

$$\Leftrightarrow \frac{h_{1\text{-}2:\mathrm{G}}(h(p))}{h(h_{1\text{-}2:\mathrm{G}}(p))} = \frac{2p_{\mathrm{Nu}} - p_{\mathrm{Nu}}^2}{p_{\mathrm{De}} + p_{\mathrm{De}}^2 - p_{\mathrm{De}}^3};$$

$$p_{\mathrm{Nu}} \equiv p+p^2-p^3;\ p_{\mathrm{De}} \equiv 2p-p^2;$$

is increasing in p. However the function is not increasing in p, as shown in figure 2 where the components are exponentially distributed with mean 1. For large t, the hazard rate for the system with component redundancy is larger (small hazard rates are preferred) than that for system redundancy. ◄

4. STANDBY REDUNDANCY AT THE COMPONENT &SYSTEM LEVELS

Assumption

5. The standby redundant components neither degrade nor fail while in standby. When inserted into the system as active components, their state is *new*. ◄

The total lifetime of the *position* (original + standby) is $T+T^*$. This is another type of redundancy: standby (*vs* active). To what extent is standby redundancy at the component level better (either in *stochastic* or *hazard rate* ordering) than standby redundancy at the system level? This section discusses some special cases, and begins with an elementary result (proof is omitted) concerning the usual stochastic order for *series & parallel* systems.

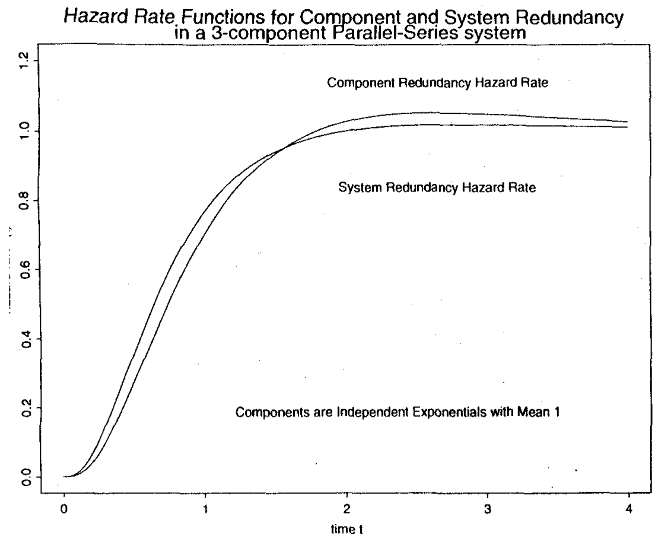

Figure 2. Hazard Rates for Component & System Redundancies in a 3-Component *Parallel-Series* System

Theorem 4

(Use assumptions 1 - 3, 5 and Notation list.)

$$\tau_{n\text{-}n:\mathrm{G}}(\boldsymbol{T}) + \tau_{n\text{-}n:\mathrm{G}}(\boldsymbol{T}^*) \stackrel{\mathrm{st}}{\leq} \tau_{n\text{-}n:\mathrm{G}}(\boldsymbol{T} + \boldsymbol{T}^*),$$

$$\tau_{1\text{-}n:\mathrm{G}}(\boldsymbol{T}) + \tau_{1\text{-}n:\mathrm{G}}(\boldsymbol{T}^*) \stackrel{\mathrm{st}}{\geq} \tau_{1\text{-}n:\mathrm{G}}(\boldsymbol{T} + \boldsymbol{T}^*). \quad ◄$$

This result says that when considering standby redundancy and stochastic ordering, redundancy at the component level is better than redundancy at the system level for *series* systems, while the reverse is true for *parallel* systems. What does this suggest about more-general k-out-of-n:G systems?

Example 2

Let T_1,T_2,T_3 and T_1^*,T_2^*,T_3^* be lifetimes of mutually s-independent components & spares which are all exponentially distributed with mean 1. Then,

$$\tau_{2\text{-}3:\mathrm{G}}(\boldsymbol{T}+\boldsymbol{T}^*) \stackrel{\mathrm{st}}{\geq} \tau_{2\text{-}3:\mathrm{G}}(\boldsymbol{T}) + \tau_{2\text{-}3:\mathrm{G}}(\boldsymbol{T}^*). \quad ◄$$

What can we say about standby component & system redundancy for the hazard rate ordering? Example 3 shows that for exponential i.i.d. r.v., and for *series* systems, standby component redundancy is better than standby system redundancy.

Example 3

Let $T_1,\ldots,T_n$, $T_1^*,\ldots,T_n^*$ be mutually s-independent exponential r.v. with mean 1. Then,

$\tau_{n\text{-}n:\mathrm{G}}(\boldsymbol{T}) + \tau_{n\text{-}n:\mathrm{G}}(\boldsymbol{T}^*) \stackrel{d}{=} \Gamma(2,n)$ — Gamma with parameters 2 and n, while

$\tau_{n\text{-}n:\mathrm{G}}(\boldsymbol{T} + \boldsymbol{T}^*) \stackrel{d}{=} \tau_{n\text{-}n:\mathrm{G}}(G_1,\ldots,G_n)$ where $G_1,\ldots,G_n$ are mutually s-independent Gamma r.v. with parameters 2,1 — $G_i \stackrel{d}{=} \mathrm{Gamma}(2,1)$ — for $i=1,\ldots,n$. Now,

$$r_{\Gamma(2,n)}(t) = \frac{n^2 t}{1+n\cdot t} \geq \frac{n\cdot t}{1+t} = n\cdot r_{\Gamma(2,1)}(t)$$

for all $t \geq 0$,

from which it follows that,

$$\tau_{n\text{-}n:\mathrm{G}}(\boldsymbol{T} + \boldsymbol{T}^*) \stackrel{\mathrm{hr}}{\geq} \tau_{n\text{-}n:\mathrm{G}}(\boldsymbol{T}) + \tau_{n\text{-}n:\mathrm{G}}(\boldsymbol{T}^*). \quad ◄$$

APPENDIX

A.1 Proof of Example 1

$$g_1(t) \equiv \Pr\{\tau(\boldsymbol{T})\vee\tau(\boldsymbol{T}^*) > t\}/\Pr\{\tau(\boldsymbol{T}\vee\boldsymbol{T}^*) > t\}$$

for any $t \geq 0$

$\tau \equiv$ lifetime of a 1-out-of-2:F system.

By demonstrating that $g_1(t)$ is eventually increasing, we show that even for this simple system, redundancy at the component level is not better in the hazard rate ordering than redundancy at the system level.

$$g_1(t) = \frac{\exp(-4t)+\exp(-2t)-\exp(-6t)}{[\exp(-2t)+\exp(-t)-\exp(-3t)]^2}$$

$$= \frac{\exp(2t)+\exp(4t)-1}{[\exp(t)+\exp(2t)-1]^2},$$

$$g'_1(t) = \leq 0 \Leftrightarrow [\exp(t) + \exp(2t)-1]\cdot[2\exp(2t)+4\exp(4t)]$$

$$- 2[\exp(t)+2\exp(2t)]\cdot[\exp(2t)+\exp(4t)-1]$$

$$= \exp(5t)[2-6\exp(-t)+2\exp(-3t)+2\exp(-4t)] \leq 0.$$

However, $\lim_{t\to\infty} [2 - 6\exp(-t) + 2\exp(-3t)+2\exp(-4t)] = 2$;

hence $g'_1(t) > 0$ for large t. *Q.E.D.*

A.2 Proof of Theorem 1

Notation

$$\prod_i \quad \prod_{i=1}^{n}.$$

We want to show that,

$$\Pr\{[\min_i(T_i)]\vee[\min_i(T_i^*)] > t\}/\Pr\{\min_i(T_i\vee T_i^*) > t\}$$

$$= \left[2\left(\prod_i \bar{F}_i(t)\right) - \left(\prod_i \bar{F}_i(t)\right)^2\right] \Big/ \prod_i [2\bar{F}_i(t) - (\bar{F}_i(t))^2],$$

is a decreasing function of t. Let $p_i = \bar{F}_i(t)$ for $i=1,\ldots,n$; it suffices to show:

$$g_2(p_1,p_2,\ldots,p_n)$$
$$\equiv \left[2\left(\prod_i p_i\right) - \left(\prod_i p_i\right)^2\right] \Big/ \prod_i (2p_i - p_i^2)$$
$$= \left[2 - \left(\prod_i p_i\right)\right] \Big/ \prod_i (2-p_i),$$

is nondecreasing in each p_i, $i = 1,\ldots,n$.

$$\frac{\partial}{\partial p_i}[g_2(\boldsymbol{p})]$$
$$= \frac{\left(\prod_i (2-p_i)\right)\cdot\left(-\prod_{j\neq i} p_j\right) - \left(2-\prod_i p_i\right)\cdot\left(-\prod_{j\neq i} (2-p_j)\right)}{\prod_i (2-p_i)^2}$$
$$= \left[\left(\prod_{j\neq i} (2-p_j)\right)\cdot\left\{-(2-p_i)\cdot\left(\prod_{j\neq i} p_j\right) + 2-\left(\prod_i p_i\right)\right\}\right] \Big/ \prod_i (2-p_i)^2$$
$$= \left[\left(\prod_{j\neq i} (2-p_j)\right)\cdot\left\{2-2\left(\prod_{j\neq i} p_j\right)\right\}\right] \prod_i (2-p_i)^2 \geq 0.$$

Hence $g_2(\boldsymbol{p})$ is nondecreasing in each p_i; the result follows. *Q.E.D.*

A.3 Proof of Corollary 1-1

View the subsystems $\max_{i\in A_j}(T_i)$, $j = 1,\ldots,m$ as m *super* components connected to each other in *series*; observe that redundancy at the component level can be viewed as redundancy at the *super* component level. The result follows from theorem 1. *Q.E.D.*

A.4 Proof of Theorem 2

Notation

$h_{1\text{-}2:\mathrm{G}}(p)$ reliability of 1-out-of-2:G system with i.i.d. components; each with reliability p
$\bar{F}(t)$ common reliability of components & spares
$r_c(t)$, $r_s(t)$ hazard rate of $[\tau(\boldsymbol{T}\vee\boldsymbol{T}^*), \tau(\boldsymbol{T})\vee\tau(\boldsymbol{T}^*)]$.

A property of k-out-of-n:G systems is that

$$p h'_{k\text{-}n:\mathrm{G}}(p)/h_{k\text{-}n:\mathrm{G}}(p)$$

is nonincreasing in p for any k,n [4].

$$\Pr\{\tau(\boldsymbol{T}\vee\boldsymbol{T}^*) > t\} = h(h_{1\text{-}2:\mathrm{G}}(\bar{F}(t)))$$

$$\Pr\{(\tau(\boldsymbol{T})\vee\tau(\boldsymbol{T}^*)) > t\} = h_{1\text{-}2:\mathrm{G}}(h(\bar{F}(t))).$$

Therefore

$$r_c(t) = \frac{h'(h_{1\text{-}2:\mathrm{G}}(\bar{F}(t)))\cdot h'_{1\text{-}2:\mathrm{G}}(\bar{F}(t))\cdot f(t)}{h(h_{1\text{-}2:\mathrm{G}}(\bar{F}(t)))}$$

$$= \left[\frac{h'(h_{1\text{-}2:\mathrm{G}}(\bar{F}(t)))}{h(h_{1\text{-}2:\mathrm{G}}(\bar{F}(t)))}\cdot h_{1\text{-}2:\mathrm{G}}(\bar{F}(t))\right] \cdot \left[\frac{h'_{1\text{-}2:\mathrm{G}}(\bar{F}(t))}{h_{1\text{-}2:\mathrm{G}}(\bar{F}(t))}\cdot\bar{F}(t)\right]\cdot\frac{f(t)}{\bar{F}(t)}$$

$$\leq \frac{h'(\bar{F}(t))}{h(\bar{F}(t))}\cdot\bar{F}(t)\cdot\left[\frac{h'_{1\text{-}2:\mathrm{G}}(\bar{F}(t))}{h_{1\text{-}2:\mathrm{G}}(\bar{F}(t))}\cdot\bar{F}(t)\right]\cdot\frac{f(t)}{\bar{F}(t)}$$

{since $h_{1\text{-}2:\mathrm{G}}(p) \geq p$ and $p\cdot h'(p)/h(p)$ is decreasing in p}

$$\leq \frac{h'(\bar{F}(t))}{h(\bar{F}(t))}\cdot\bar{F}(t)\cdot\frac{h'_{1\text{-}2:\mathrm{G}}(h(\bar{F}(t)))}{h_{1\text{-}2:\mathrm{G}}(h(\bar{F}(t)))}\cdot h(\bar{F}(t))\cdot\frac{f(t)}{\bar{F}(t)}$$

{since $h(p) \leq p$ and $p\cdot h'_{1\text{-}2:\mathrm{G}}(p)/h_{1\text{-}2:\mathrm{G}}(p)$ is decreasing in p}

$$= r_s(t).$$

This proves the result when $\bar{F}(t)$ has a pdf. A standard argument 'involving convoluting $\bar{F}$ with a s-normal r.v. with mean 0 and variance $1/n$ and then taking limits' extends the result where a pdf does not exist. *Q.E.D.*

A.5 Proof of Theorem 3

Notation

q $1-p$
m $n-1$.

It suffices to show that

$$g_3(p) \equiv \frac{h_{1\text{-}2:\mathrm{G}}(h_{2\text{-}n:\mathrm{G}}(p))}{h_{2\text{-}n:\mathrm{G}}(h_{1\text{-}2:\mathrm{G}}(p))} = \frac{2h_{2\text{-}n:\mathrm{G}}(p) - [h_{2\text{-}n:\mathrm{G}}(p)]^2}{h_{2\text{-}n:\mathrm{G}}(2p-p^2)},$$

is increasing in p.

$$h_{2\text{-}n:\mathrm{G}}(p) = 1 - (m+1)\cdot q^m + m\cdot q^{m+1},$$

$$h_{2\text{-}n:\mathrm{G}}(2p-p^2) = 1-(m+1)\cdot q^{2m} + m\cdot q^{2(m+1)}.$$

Therefore,

$$g_3(p) = \frac{1-(m+1)^2\cdot q^{2m} - m^2\cdot q^{2(m+1)} + 2m\cdot(m+1)\cdot q^{2m+1}}{1-(m+1)\cdot q^{2m} + m\cdot q^{2m+2}},$$

$g'_3(p) \ge 0 \Leftrightarrow$

$$2[1-(m+1)\cdot q^{2m} + m\cdot q^{2m+2}]\cdot[(m+1)\cdot q^m - m\cdot q^{m+1}]$$
$$\cdot[m\cdot(m+1)\cdot q^{m-1} - m\cdot(m+1)\cdot q^m]$$
$$\ge [1-(m+1)^2\cdot q^{2m} - m^2\cdot q^{2(m+1)} + 2m\cdot(m+1)\cdot q^{2m+1}]$$
$$\cdot[2m\cdot(m+1)\cdot q^{2m+1} - 2m\cdot(m+1)\cdot q^{2m+1}]$$
$$\Leftrightarrow (1-q)\cdot[(m+1)-m\cdot q]\cdot[1-(m+1)\cdot q^{2m} + m\cdot q^{2(m+1)}]$$
$$\ge (1-q^2)\cdot[1-(m+1)^2\cdot q^{2m} - m^2\cdot q^{2(m+1)}$$
$$+ 2m\cdot(m+1)\cdot q^{2m+1}]$$
$$\Leftrightarrow m\cdot(1-q^{2(m+1)}) \ge (m+1)\cdot[q - q^{2m+1}]$$
$$\Leftrightarrow \frac{m}{1+m} \ge \frac{q-q^{2m+1}}{1-q^{2(m+1)}} \equiv g_4(q)$$

Since $\lim_{q\to 1}[g_4(q)] = m/(1+m)$, it suffices to show that $g_4(q)$ is an increasing function of q.

$$g'_4(q) \ge 0$$
$$\Leftrightarrow 1+(2m+1)\cdot q^{2m+2} - (2m+1)\cdot q^{2m} - q^{4m+2} \ge 0$$
$$\Leftrightarrow (1-q^2)\cdot[1+q^2+\ldots+(q^2)^{2m}] \ge (2m+1)\cdot q^{2m}\cdot(1-q^2)$$
$$\Leftrightarrow [1+x+\ldots+x^{2m}] \ge (2m+1)\cdot x^m \text{ for } x = q^2 \in [0,1]$$
$$\Leftrightarrow [1+x+\ldots+x^{2m}]/(2m+1) \ge x^m$$

which is true since the arithmetic mean of $(1,x,\ldots,x^{2m})$ exceeds its geometric mean. *Q.E.D.*

A.6 Proof of Example 2

It suffices to demonstrate that

$$\Pr\{\tau_{2\text{-}3:G}(T) + \tau_{2\text{-}3:G}(T^*) > t\} \le \Pr\{\tau_{2\text{-}3:G}(T + T^*) > t\}$$
$$\Leftrightarrow \exp(-2t)\cdot[18t - 27] + \exp(-3t)\cdot[28 + 12t]$$
$$\le \exp(-2t)\cdot[(1+t)^2\cdot(3-2\exp(-t) - 2t\cdot\exp(-t))]$$

for all $t > 0$.

This in turn is equivalent to showing,

$$3t^2 - 12t + 30 - \exp(-t)\cdot[28+12t + 2(1+t)^3] \ge 0$$

for all t,

which is achieved through successive differentiation. This example shows that for 2-out-of-3:G systems with i.i.d. exponential components & spares, standby redundancy at the component level is stochastically better than standby redundancy at the system level.

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