Indirect Assessment of System Reliability

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***Abstract*—Assessing system-reliability has been a major goal of many people in many walks of life, including those that develop and deploy systems. For that purpose many methods and techniques have been published. However, most of these techniques and methods rely heavily on failure data. Because the design of high-reliability systems generally requires that the individual system components have extremely high reliability, even after long periods of time, and because short product-development times and tightened-budgets imply that reliability tests must be conducted with severe time-constraints, frequently no failures occur during such tests. Thus, there is a growing need to assess reliabilities for complex systems for which it is difficult to collect failure data.**

This paper focuses on a method that expresses the failure time and consequently the reliability of a system as a function of several explanatory variables (referred to as covariates). Then, the system reliability is assessed indirectly by using information on these covariates. From a practical viewpoint, this method might be more complicated and difficult to apply. The author still recommends this methodology because there are many situations where it is difficult to have data on failure times of expensive and complex systems, and this method can handle such situations.

***Index Terms*—Cumulative damage model, diffusion process, hitting time, Ito process, log-normal process, maximum likelihood estimate, stochastic process, stress and strength system.**

ACRONYMS[1]

iid	s-independent and identically distributed
inf	infimum (greatest lower bound of a set)
MLE	maximum likelihood estimate
pdf	probability density function
r.v.	random variable
s-	implies: statistical(ly).

NOTATION

T	failure time of a system
$X_i(t)$	value of covariate i at time t, $i = 1, \ldots, k$
A_i	permissible set for covariate i, $i = 1, \ldots, k$
$\overline{A}_i$	complement of A_i, $i = 1, \ldots, k$
t_0	0
$X_i^*(t)$	stopped process:

$$\begin{cases} X_i(t), & t \le T \\ X_i(T), & t > T, \end{cases} \qquad i = 1, \ldots, k$$

$S(t)$	$\Pr\{T > t\}$: reliability function
z	threshold
m	number of unknown parameters
θ	$(\theta_1, \ldots, \theta_m)$: unknown vector-parameter
θ_2^*	initial value for θ_2
$\hat{\theta}$	$(\hat{\theta}_1, \ldots, \hat{\theta}_m)$: MLE of θ
$B(t)$	standard Brownian motion
x_0	$X_1(0)$
β	$\theta_1 - (\theta_2/2)$
β^*	initial value for β
$K_r(s, x, r, z; \theta)$	$\Pr\{T \le r \mid X(s) = x, \theta\}$
$f(s, x, t, z; \theta)$	conditional pdf$\{T \mid X_1(s) = x\}$
$p(s, x, t, y; \theta)$	transition pdf$\{X_1(t) \mid X_1(s) = x\}$
$q(s, x, t, y; z, \theta)$	transition pdf$\{X_1^*(t) \mid X_1^*(s) = x\}$
$S(t \mid x_0, \theta)$	$\Pr\{T > t \mid X(0) = x_0, \theta\}$.

I. INTRODUCTION

THE SIMPLEST notion in reliability theory is the failure time of a system; the main problem connected to the failure time is that of finding its distribution and consequently assessing the system reliability. The traditional approach is that the distribution is estimated from the experimental data on the failure times of several s-identical copies of the system [8], [10]. Unfortunately, however, with high reliability systems and with short development times, reliability tests must be conducted with severe time constraints, and frequently no failures occur during such test. Thus, it is impossible to assess reliability with traditional life-tests that record failure times.

To circumvent such difficulties, the failure time is defined as a function of several covariates. For example, thinking of degradation as a covariate, there are situations where system degradation measures can be taken over time and where a relationship between system-failure and amount-of-degradation makes it possible to use degradation models and data to make inferences and predictions about failure time. Usually, degradation models begin with a deterministic description of the degradation process that is often in the form of a differential equation or system of differential equations. Randomness can be introduced using probability distributions to describe variability in initial conditions and model parameters. To illustrate, let $D(t)$ be the amount of automobile tire-tread wear at time t, and wear rate be $dD(t)/dt = \beta_1$, then $D(t) = D(0) + \beta_1 \cdot t$. Here, $D(0)$ and β_1 are treated as r.v. In general, if $D(t; \beta_1, \ldots, \beta_k)$ denotes the degradation level of a system at time t, where more than one of the $\beta_1, \ldots, \beta_k$ is random; then, the failure time can be the first time that $D(t; \beta_1, \ldots, \beta_k)$ crosses the known threshold. For various degradation models and more discussion about the relationship between degradation-level and system-failure see [10] and the references cited there.

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[1]The singular and plural of an acronym are always spelled the same.

Reference [2] proposed the model, commonly referred to as the Cox proportional hazards model, in which the hazard function is expressed in terms of covariates. In spite of its appealing features, to assess system reliability under this model still requires failure data. Reference [9], in a series of articles, have proposed models that express either the survival function or the failure time of a system in terms of covariates. However, no statistical methods are discussed for identifying the parameters of the reliability function. For an extensive list of references, see [9], [14]. This paper gives a general framework for modeling failure time as well as describing a general methodology for estimating the parameters of a reliability function.

To express the failure time as a function of several covariates as well as to assess the reliability of a system, there are 3 separate problems:

1) Choosing appropriate stochastic models for covariates,

2) Expressing the system reliability in terms of covariates,

3) Identifying the parameters of the reliability function by statistical means.

This approach has 2 advantages.

1) The reliability of highly reliable systems can be assessed without having failure data.

2) It can provide effective information regarding survival of a system while it is in operation.

Section II develops general formulation and describes the methodology. Section III illustrates the technique using specific stochastic model.

II. General Formulation

Let the failure time be defined in terms of k covariates. Since, in many practical situations, the covariates that are commonly identified have been observed to display substantial variation, the stochastic process is a reasonable way to model their behaviors. For example, the status of a stress–strength system clearly changes dynamically with time. The dynamic approach to modeling a stress–strength system accommodates a time-dependent stress, $V(t)$, and a strength, $W(t)$. One can formulate this problem in two ways; it is much easier to work with the second way.

1) Treat $V(t)$ and $W(t)$ as two covariates ($k = 2$).

2) Define $X_1(t) = V(t) - W(t)$, the difference between stress and strength at time t; and treat this process as one covariate ($k = 1$).

In general, the crux of modeling the values of the covariates is the quantification of the behavior of the random time points such as the instance at which the process $X_1(t)$ in the stress–strength system goes above zero. Such critical moments are often termed as hitting-times of the stochastic processes.

Consider the hitting times:

$$T_i(A_i) = \mathrm{Inf}[0 \le t \le \infty\colon X_i(t) \in \overline{A}_i]. \tag{1}$$

The random time in (1) is the first time when covariate i of a system is not within permissible limits. In this setting, the failure time of a system is

$$T = \min[T_1(A_1), \ldots, T_k(A_k)]. \tag{2}$$

For the stress–strength system $A_1 = \{x\colon x < 0\}$ and $T = \mathrm{Inf}[0 \le t \le \infty\colon X_1(t) \ge 0]$. Reference [5] assumes the process, $X_1(t)$, is Brownian motion and estimates the reliability function for the stress–strength system. Brownian motion, with positive drift, was also used to model system wear and tear: system fails whenever the wear and tear reaches a certain critical threshold [9].

A special case of (1) and (2) also provides a useful model of cumulative damage that can be used to study the reliability of various systems. For example, let

$N(t) \equiv$ the number of shocks to a system during $(0, t]$,

$Y_i \equiv$ the damage from shock i inflicted upon the system.

If the system fails when the total damage exceeds a known threshold z, then

$$T = \mathrm{Inf}\left[0 \le t \le \infty;\ X_1(t) = \sum_{i=1}^{N(t)} Y_i \ge z\right].$$

For various damage models see [4], [6].

Equations (1) and (2) can also be used for a series system with k components, and $T_i(A_i)$ is the failure time of component i, $i = 1, \ldots, k$. T is the failure time of the system.

Frequently, engineers are interested in the reliability of a system over a specific time period, say $(0, s]$. It follows from (2) that the system reliability is

$$S(s) = \Pr\{T_1(A_1) > s, \ldots, T_k(A_k) > s\}. \tag{3}$$

Consider the case: $k = 1$. Let the expression for $S(s)$ be known, except for several unknown parameters, $\theta_1, \ldots, \theta_m$. If there are observations from $X_1(t)$ on a discrete time-grid, then the likelihood function is known and is used to estimate the unknown parameters. However, in this case, it is impossible to observe the process $\{X_1(t);\ t \ge 0\}$ after T; i.e., there are only observations from the killed or stopped process $X_1^*(t)$ on a discrete time-grid. This restriction on the values of $X_1(\cdot)$ is a consequence of the role of T as a time at which the process is killed, so that it is not observable after that time. For example for a stress–strength system, the strength process $W(t)$ is not observable after T. Thus, the difference between stress and strength $X_1(t)$ is not observable after this time.

If there are observations from $X_1^*(t)$ on a discrete time grid, then the likelihood function can be based on these observations and used to estimate the unknown parameters $\theta_1, \ldots, \theta_m$. However, to write the likelihood function, one must know the distribution of $X_1^*(t)$, which can be complicated. The complication in the distributional property of $\{X_1^*(t);\ t \ge 0\}$ stems from the restriction that $X_1^*(t)$ is known to be in A_1 for $0 \le t < T$. At $t = T$, $X_1^*(t)$ is absorbed at a point that belongs to $\overline{A}_1$. For example, for a stress–strength system, $X_1^*(t)$ is known to be positive for $0 \le t < T$, and for $t \ge T$, $X_1^*(t)$ is absorbed at 0. The following paragraph describes a technique for obtaining the distributional property of the $X_1^*(t)$ when A_1 is an interval.

Let $A_1 = (-\infty, z)$; then, $T = \mathrm{Inf}[0 < t < \infty\colon X_1(t) \ge z]$. Based on discrete observations, $X_1^*(t_i)$, $i = 1, \ldots, n$ from the stopped process $X_1^*(t)$, and given the likelihood function for this stopped process, then the unknown parameters and $S(t|x_0, \theta)$ can be estimated.

In general, the transition pdf of $\{X_1^*(t);\ t \geq 0\}$, which is related to $p(s, x, t, y;\ \theta)$ of the original process $\{X_1(t);\ t \geq 0\}$, is

$$q(s, x, t, y;\ \theta, z) = p(s, x, t, y;\ \theta) - \int_s^t p(r, z, t, y;\ \theta) \cdot dK_r(s, x, r, z; \theta), \qquad s < t, \cdot \max[x, y] < z. \tag{4}$$

Let $y = z$, then $\Pr\{X^*(t) = z | X(s) = x\} = f(s, x, t, z;\ \theta)$. Let $x = y = z$, then $\Pr\{X^*(t) = z | X(s) = z, \theta\} = 1$.

To estimate θ, from discrete observations of $X_1^*(t)$ at time points $0 < t_1 < \cdots < t_n$, the log-likelihood function is

$$\begin{aligned}\ell_n(\theta) &= \sum_{i=1}^{n} \log\left[q\left(t_{i-1}, X_1^*(t_{i-1}), t_i, X_1^*(t_i);\ \theta\right)\right] \\ &= \sum_{i=1}^{n} \log\Bigg[p\left(t_{i-1}, X_1^*(t_{i-1}), t_i, X_1^*(t_i);\ \theta\right) \\ &\qquad - \int_{t_{i-1}}^{t_i} p\left(r, z, t_i, X_1^*(t_i);\ \theta\right) \\ &\qquad \cdot dK_r\left(t_{i-1}, X_1^*(t_{i-1}), r, z;\ \theta\right)\Bigg].\end{aligned} \tag{5}$$

$X_1^*(t_i) < z,\ i = 1, \ldots, n$.

The MLE of θ can be obtained by solving the likelihood equations

$$\frac{\delta \ell_n(\theta)}{\delta \theta_i} = 0, \qquad i = 1, \ldots, m. \tag{6}$$

Estimate $S(t|x_0, \theta)$ by $S(t|x_0, \hat{\theta})$.

In this situation, one need not monitor the $X_1^*(t)$ until time T. One only needs to observe $X_1^*(t)$ at several time points.

Estimating the reliability function in the 1-case can be extended to the multi-dimensional case of $k > 1$, provided that one can model all the processes $X_i(t),\ i = 1, \ldots, k$ and $X_i^*(t),\ \ i = 1, \ldots, k$ jointly. However, in practice, such information might not be available or might be difficult to obtain. In such situations, one can seek the bound for the system reliability if some information about the dependence structure of $T_1(A_1), \ldots, T_k(A_k)$ is available. For example, if $T_1(A_1), \ldots, T_k(A_k)$ are known to have the positively orthant dependence structure:

$$\Pr\{T_1(A_1) > t_1, \ldots, T_k(A_k) > t_k\} \geq \prod_{i=1}^{k} \Pr\{T_i(A_i) > t_i\};$$

then

$$S(s) \geq \prod_{i=1}^{k} \Pr\{T_i(A_i) > s\}.$$

Let it be desirable to check whether or not the reliability meets a given specification. If the lower bound, $\prod_{i=1}^{k} \Pr\{T_i(A_i) > s\}$, in the lower bound in this example already meets or exceeds the specification, then one knows for sure that the system meets the specification. Such conclusions are gratifying, particularly when the joint modeling of $X_1(t), \ldots, X_k(t)$ is not possible. For more details about the dependence structure of hitting-times of a multivariate stochastic process see [3] and references cited there. One can estimate $\Pr\{T_i(A_i) > s\}$, $i = 1, \ldots, k$, separately by using a technique described for $k = 1$.

III. THE MODEL AND ESTIMATORS

Notation

h	unit of time
$X_1(n)$	physically observable system-state at nh, $n = 1, 2, \ldots$
$\{\epsilon_n\}$	iid r.v.

The system wear is defined as

$$X_1(n+1) - X_1(n) = \sigma(X_1(n)) \cdot \epsilon_n + \mu(X(n)) \cdot h.$$

Thus if the system's initial wear is x, then over h units of time its wear increases on the average by $(\sigma(x)) \cdot E[\epsilon_n] + (\mu(x)) \cdot h$ and the standard deviation of the average increase in wear is $(\sigma(x)) \cdot (\mathrm{Var}[\epsilon_n])^{1/2}$. Now, if h can decrease to 0, and if $\{\epsilon_n\}$ are Gaussian r.v. with the common-mean 0 and common-variance 1, then the resulting stochastic differential equation is

$$dX_1(t) = \sigma(X_1(t)) dB(t) + [\mu(X_1(t))]\ dt.$$

For more details see [14]. Now, let $X_1(t)$ be described as a unique solution to the following nonlinear stochastic differential equation

$$dX_1(t) = (\theta_1 \cdot X_1(t))\, dt + \left(\sqrt{\theta_2} \cdot X_1(t)\right) dB(t). \tag{7}$$

That is, the variance as well as anticipated change are proportional to instantaneous size. This is a "Homogeneous log-normal Ito process" or "Diffusion process." It is nonnegative and exhibits long-run exponential growth (or decay). Equation (7) is often used to model prices of assets, e.g., shares of stock that are traded in a perfect market. It is also used in describing system-wear and other physical phenomena; [7], [12], [14] provide a comprehensive review of the Ito process and its applications. The model (7) is but one of many models encountered in practice. Equation (7) is used here solely to illustrate the approach.

Let the failure time of the system be

$$T = \mathrm{Inf}[0 \leq t \leq \infty\colon X_1(t) \geq z]. \tag{8}$$

To estimate the system reliability, first obtain the MLE of θ. In this case, $m = 2$. From [13], the transition pdf of $X_1(t)$ and the conditional pdf of T, given $X_1(s) = x$ are

$$p(s, x, t, y;\ \theta) = \frac{1}{y \cdot \sqrt{2\pi \cdot \theta_2 (t-s)}} \cdot \exp\left[-\left(\frac{1}{2\theta_2(t-s)}\right) \cdot (\log(y) - \log(x) - \beta(t-s))^2\right], \tag{9}$$

$$f(s, x, t, z;\ \theta) \equiv \frac{\log(z/x)}{\sqrt{2\pi \cdot \theta_2 (t-s)^{3/2}}} \cdot \exp\left[-\frac{[\log(z/x) - \beta \cdot (t-s)]^2}{2\theta_2(t-s)}\right], \tag{10}$$

TABLE I
POINT ESTIMATES OF θ_1, θ_2

	$\theta_1 = 0$,	$\theta_2 = 1$	$\theta_1 = 1$,	$\theta_2 = 1$	$\theta_1 = 1$,	$\theta_2 = 2$
$\Delta = 1/2$	$\hat{\theta}_1 = 0.15$,	$\hat{\theta}_2 = 1.23$	$\hat{\theta}_1 = 1.13$,	$\hat{\theta}_2 = 0.58$	$\hat{\theta}_1 = 1.21$,	$\hat{\theta}_2 = 2.10$
$\Delta = 1/4$	$\hat{\theta}_1 = 0.17$,	$\hat{\theta}_2 = 1.37$	$\hat{\theta}_1 = 1.32$,	$\hat{\theta}_2 = 0.70$	$\hat{\theta}_1 = 1.61$,	$\hat{\theta}_2 = 2.61$
$\Delta = 1/8$	$\hat{\theta}_1 = 0.25$,	$\hat{\theta}_2 = 1.58$	$\hat{\theta}_1 = 1.50$,	$\hat{\theta}_2 = 0.78$	$\hat{\theta}_1 = 1.78$,	$\hat{\theta}_2 = 2.92$
$\Delta = 1/16$	$\hat{\theta}_1 = 0.36$,	$\hat{\theta}_2 = 1.92$	$\hat{\theta}_1 = 1.80$,	$\hat{\theta}_2 = 0.85$	$\hat{\theta}_1 = 2.11$,	$\hat{\theta}_2 = 3.32$

Also, by using (4), the transition pdf of $X_1^*(t)$ is

$$\begin{aligned} q(r,\, s,\, t,\, x,\, y;\; z,\, \theta) &= \frac{\exp(-\Psi_1(s,\, t,\, x,\, y;\; \theta))}{y\sqrt{2\pi\theta_2\cdot(t-s)}} \\ &\quad - \int_s^t \frac{\exp(-\Psi_2(r,\, t,\, x,\, y;\; z,\, \theta))}{y\sqrt{2\pi\theta_2\cdot(t-r)}} \\ &\quad \cdot \frac{\log\left(\frac{z}{x}\right)\cdot\exp(-\Psi_3(r,\, s,\, x;\; z,\, \theta))}{\sqrt{2\pi\theta_2\cdot(r-s)^{3/2}}}\, dr. \end{aligned} \tag{11}$$

$$\Psi_1(s,\, t,\, x,\, y;\; \theta) \equiv \frac{\left(\log\left(\frac{y}{x}\right) - \beta\cdot(t-s)\right)^2}{2\theta_2\cdot(t-s)},$$

$$\Psi_2(r,\, t,\, x,\, y;\; z,\, \theta) \equiv \frac{\left(\log\left(\frac{y}{z}\right) - \beta\cdot(t-r)\right)^2}{2\theta_2\cdot(t-r)},$$

$$\Psi_3(r,\, s,\, x,\, z,\, \theta) \equiv \frac{\left(\log\left(\frac{z}{x}\right) - \beta\cdot(r-s)\right)^2}{2\theta_2\cdot(r-s)}.$$

From (5) the actual log-likelihood function is

$$\ell_n(\theta) = \sum_{i=1}^{n} \log\left(q(t_{i-1},\, x_{i-1}^*,\, t_i,\, x_i^*;\; z,\, \theta)\right). \tag{12}$$

q is given by (11).

Here, $x_i^* < z$, $i = 1, \ldots, n$.

The MLE of β and θ_2 can be obtained by solving the nonlinear equations:

$$\frac{\delta\ell_n(\theta_2,\, \beta)}{\delta\beta} = 0, \tag{13}$$

$$\frac{\delta\ell_n(\theta_2,\, \beta)}{\delta\theta_2} = 0. \tag{14}$$

Because (13) and (14) cannot be solved analytically for $\hat{\beta}$ and $\hat{\theta}_2$ some numerical method must be used. Solutions of (13) and (14) with an iterative procedure such as Newton–Raphson can be used to obtain $\hat{\theta}_2$ and $\hat{\beta}$. The initializing values of θ_2 and β in such a scheme can be obtained by pretending that the observations $x_1^*, \ldots,$ and x_n^* are from the original process $X_1(t)$ and thus replacing q by p in (12). Here, p is given by (9). By solving (12) and (13), the solutions are

$$\beta^* = \frac{\sum_{i=1}^{n} \log\left(\frac{x_{i+1}^*}{x_{i-1}^*}\right)}{\sum_{i=1}^{n} (t_i - t_{i-1})},$$

$$\theta_2^* = \frac{1}{n}\cdot\sum_{i=1}^{n}\left[\frac{\log\left(\frac{x_i^*}{x_{i-1}^*}\right) - \beta\cdot(t_i - t_{i-1})}{t_i - t_{i-1}}\right]^2.$$

When the likelihood function is not well behaved or when even "first derivatives of the log-likelihood with respect to unknown parameters" are inconvenient to compute, a direct search procedure for finding the MLE can be the best approach. The simplex search procedure [11] has proved successful in many problems, particularly when there are not too many parameters present. Other search procedures have been presented in the literature. For more details about such procedures and how to implement them, see [1].

Using $\hat{\beta}$ and $\hat{\theta}_2$, one can obtain the MLE of θ_1. Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be the MLE of θ_1 and θ_2, respectively; then the reliability function $S(t|x_0,\, \theta)$ can be estimated by

$$\hat{S}(t|x_0,\, \hat{\theta}) = \int_t^\infty f(0,\, x_0,\, x,\, z;\; \hat{\theta})\, dx. \tag{15}$$

f is given by (10).

As an illustration, use the data generated from the model (7), plugging in known values of θ_1 and θ_2. The data are then truncated at a known value of z. The numerical results arising from 12 sets of simulated data are given. The Appendix contains the full description of this simulation. Here, let $z = 25$ and $n = 20$ for all cases. Also, let the data be equally spaced:

$$\begin{aligned} &t_i - t_{i-1} = \Delta, \qquad i = 1, \ldots, 20, \\ &X^*(0) = 0, \qquad t_0 = 0. \end{aligned}$$

The results are in Table I. The estimators agree well with actual values. However, sometimes there is a larger difference between the actual value of the parameter and its MLE. This discrepancy between the estimate and the actual value can be a consequence of Δ being large. Table I shows that, as Δ gets smaller the difference between the actual value of the parameter and its MLE gets larger. This makes sense, because larger Δ implies that the process $X_1^*(t)$ is being monitored longer.

Now, given $\hat{\theta}_1$ and $\hat{\theta}_2$ in Table I, the system reliability of the system can be estimated using (15). One can assess system reliability for which failure data are not available by monitoring the process $X_1^*(t)$.

APPENDIX

This describes the way 20 observations were generated from the model (7) with $\theta_1 = 0$, $\theta_2 = 1$, and $\Delta = 1/2$. A similar procedure was used to generate observations for various values of θ_1, θ_2, and Δ. The IMSL standard Gaussian random number generator was used with ISEED $= 123\,457$ to generate 20 observations from the standard Gaussian distribution. The IMSL routine RNSET was used to initialize the seed of the random number generator. For more details, see [15].

Denoted $X_1(t_i) = X_1(\Delta)$ by $X(i)$, $i = 1, \ldots, n$, then $X(j+1)$ was obtained from $X(j)$ by using

$$X(j+1) = X(j) \cdot (1 + X(j)) \cdot (1/2)^{1/2}$$

[the $(j+1)$ random number], $j = 0, \ldots, 19$. Here, $X(0) = 1$.

This procedure results in 20 observations from the model (7) with $\theta_1 = 0$, $\theta_2 = 1$, and $\Delta = 1/2$: $X(1), \ldots, X(20)$. To obtain 20 observations from $X_1^*(t)$, define, for $j = 1, \ldots, 20$,

$$X^*(j) \equiv x(j) \text{ if } X(j) < 25:$$
$$X^*(j) \equiv 25 \text{ if } X(j) \geq 25.$$

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