Calculation of Acoustic Loading on Transducers in the Time Domain

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Abstract **One technique that has been widely used to calculate the acoustic loading on single transducers and transducer elements of an array is the numerical solution of the Helmholtz integral equation. This technique calculates the acoustic pressure on the radiating surface in the frequency domain. When working with nonlinear devices or when employing short pulses, it is often more convenient to work in the time domain. In this paper we show how the Kirchhoff integral equation can be used to calculate the acoustic loading on a transducer in the time domain. This integral equation expresses the pressure at a given position and time in terms of past values of the pressure and its time derivative over the surface. For a given spatial discretization the numerical solution of this equation becomes unstable if the time step is too small. We present a method for stabilizing the solution as the time steps become small. Numerical results will be presented and compared with known analytic solutions and with transformed frequency-domain results.**

I. INTRODUCTION

It is well known that acoustic loading can significantly affect the performance of a transducer. This is especially true when the transducer is part of an array. For a few simple geometries one can obtain analytic expressions for this acoustic loading. In most cases, however, the acoustic loading is obtained by numerically approximating the underlying differential or integral equations. One common technique is to numerically solve the Helmholtz integral equation on the radiating surface. This integral equation relates the surface pressure and normal velocity or acceleration in the frequency domain. The numerical solution of this equation reduces to a system of linear equations that must be solved at each frequency.

For problems involving nonlinear transducers or very short pulses it is often more convenient to work in the time domain. The loading in the time domain can be obtained by numerically transforming frequency domain results computed at a large number of frequencies. There are, however, methods that work directly in the time domain. One such method is the numerical solution of the Kirchhoff integral equation. This integral equation is the Fourier transform of the Helmholtz integral equation. It expresses the surface pressure at a given position and time in terms of past values of pressure and the time derivative of pressure over the surface as well as the history of the normal acceleration over the surface. The numerical solution of this equation reduces to a recursion relation in time as opposed to solutions of systems of linear equations as was the case in the frequency domain. One of the first references to this approach was a paper by Mitzner [1]. He used the Kirchhoff integral equation to solve for the rigid scattering from an arbitrary shaped surface. A number of other authors have also used this approach. One of the biggest problems with this approach has been the instability of the recursion relation for small time steps [2][3]. In this paper we will present a method for stabilizing the procedure when the time steps are small.

It is well known that the solution of the Helmholtz integral equation is nonunique at certain frequencies corresponding to resonances of the associated interior problem [4]. It will be shown here that these frequencies appear in the error of the time-domain solution and that this portion of the error does not decay versus time. Computed results in the time domain will be presented and compared with analytical solutions for several example problems.

II. BASIC EQUATIONS

The Helmholtz integral equation in the frequency domain can be written

$$\tfrac{1}{2}P(\zeta,\omega) - \int_S P(\xi,\omega)\frac{\partial G}{\partial n_\xi}(\zeta,\xi,\omega)\,dS(\xi) = \rho\int_S G(\zeta,\xi,\omega)a(\xi,\omega)\,dS(\xi) \quad (1)$$

where G is the free-space Green's function given by

$$G(\zeta,\xi,\omega) = \frac{e^{-ikr}}{4\pi r} = \frac{e^{-i\omega r/c}}{4\pi r} \quad (2)$$

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with normal derivative

$$\frac{\partial G}{\partial n_\xi} = -\frac{e^{-i\omega r/c}}{4\pi r^2}(1 + i\omega r/c)\frac{\partial r}{\partial n_\xi}. \tag{3}$$

Here S is the radiating surface, r is the distance between the surface points ζ and ξ, n_ξ is the unit normal to the surface at ξ, ρ is the fluid density, c is the sound speed, ω is the angular frequency, $P(\cdot,\omega)$ is the Fourier transform of the surface pressure, and $a(\cdot,\omega)$ is the Fourier transform of the normal acceleration on the surface.

The Fourier transform provides a correspondence between delays in the time domain and complex exponential multipliers in the frequency domain. In particular,

$$p(t-\tau) \sim e^{-i\omega\tau}P(\omega). \tag{4}$$

Thus, taking the inverse Fourier transform of the Helmholtz integral equation leads to the Kirchhoff integral equation

$$\tfrac{1}{2}p(\zeta,t) = \rho\int_S \frac{1}{4\pi r}a(\xi,t_r)\,dS - \int_S \frac{1}{4\pi r^2}\frac{\partial r}{\partial n_\xi}\big[p(\xi,t_r) + \frac{r}{c}\,\dot p(\xi,t_r)\big]\,dS \tag{5}$$

where t_r is the retarded time $t_r = t - r/c$. It is this equation that will serve as the basis of our numerical procedure. Moreover, it is easily shown that

$$\frac{\partial r}{\partial n_\xi} = \frac{(\xi-\zeta)\cdot n(\xi)}{r}. \tag{6}$$

III. NUMERICAL APPROXIMATIONS

In this section we describe the approximations to the Kirchhoff integral equation that lead to our numerical procedure. The approach used is very similar to that used by Mitzner [1]. Suppose that the surface S is subdivided into K small sub areas $\{S_k\}$. We will assume that for any fixed time t the pressure p, its time derivative $\dot p$, and the normal acceleration a are approximately constant over each sub area S_k. We will denote these constant values by $p_k(t)$, $\dot p_k(t)$, and $a_k(t)$ respectively. However, we do not assume that the delays r/c are constant over each sub area. Evaluating (5) at a reference point ζ_j on each sub area S_j and using the above assumptions, we get

$$\tfrac{1}{2}p_j(t) = \rho\sum_{k=1}^K \int_{S_k}\frac{1}{4\pi r_j}a_k(t-r_j/c)\,dS - \sum_{k=1}^K\int_{S_k}\alpha_j(\xi)\big[p_k(t-r_j/c) + \frac{r_j}{c}\,\dot p_k(t-r_j/c\big]\,dS. \tag{7}$$

where

$$\alpha_j(\xi) = \frac{1}{4\pi r_j^2}\frac{(\xi-\zeta_j)\cdot n(\xi)}{r_j}. \tag{8}$$

We assume that each sub area S_k is defined by a smooth one-to-one mapping from a two-dimensional base region into three-dimensional space. We generally take the base region to be a rectangle. The integration over S_k involves choosing quadrature points in the base region and mapping these onto the surface. The integrals over S_k are approximated using a quadrature formula of the form

$$\int_{S_k} f(\xi)\,dS(\xi) \doteq \sum_{l=1}^L w_{kl} f(\xi_{kl})\,dS(\xi_{kl}). \tag{9}$$

Here ξ_{kl} is the image of the l-th quadrature point under the mapping defining the surface region S_k, $dS(\xi_{kl})$ is the surface Jacobian of this mapping evaluated at the quadrature point, and w_{kl} is the quadrature weight corresponding to ξ_{kl}. We generally use two-dimensional Gaussian quadrature over the base rectangles. Using this quadrature formula in (7), we obtain the approximate relation

$$\tfrac{1}{2}p_j(t) = \rho\sum_{k=1}^K\sum_{l=1}^L \Phi_{jkl}a_k(t-\hat t_{jkl}) - \sum_{k=1}^K\sum_{l=1}^L \Omega_{jkl}\big[p_k(t-\hat t_{jkl}) + \hat t_{jkl}\,\dot p_k(t-\hat t_{jkl})\big] \tag{10}$$

where

$$\Omega_{jkl} = \frac{w_{kl}\,dS(\xi_{kl})}{4\pi r_{jkl}^2}\frac{(\xi_{kl}-\zeta_j)\cdot n(\xi_{kl})}{r_{jkl}}, \tag{11}$$

$$\Phi_{jkl} = \frac{w_{kl}\,dS(\xi_{kl})}{4\pi r_{jkl}}, \tag{12}$$

r_{jkl} is the distance between ζ_j and ξ_{kl}, and $\hat t_{jkl} = r_{jkl}/c$.

We now choose a time step Δt. The delay $\hat t_{jkl}$ is in general not an integer multiple of the time step Δt. Let

$$\hat t_{jkl} = n_{jkl}\Delta t + \gamma_{jkl}\Delta t \qquad \text{with } 0 \le \gamma_{jkl} < 1 \tag{13}$$

where n_{jkl} is an integer. We make the following linear approximation at the m-th time step $t_m = m\Delta t$

$$p_k(t_m - \hat t_{jkl}) = p_k(t_m - n_{jkl}\Delta t - \gamma_{jkl}\Delta t) \tag{14}$$

$$\doteq (1-\gamma_{jkl})\,p_k^{(m-n_{jkl})} + \tag{15}$$

$$\gamma_{jkl}\,p_k^{(m-n_{jkl}-1)}. \tag{16}$$

In this equation $p_k^{(n)}$ denotes the value of p_k at the n-th time step. The quantities $\dot p_k$ and a_k are approximated

similarly. With these approximations (10) becomes

$$\begin{aligned}\tfrac{1}{2}p_j^{(m)} =&\rho\sum_{k=1}^{K}\sum_{l=1}^{L}\Phi_{jkl}\Big\{(1-\gamma_{jkl})a_k^{(m-n_{jkl})}+\\ &\gamma_{jkl}\,a_k^{(m-n_{jkl}-1)}\Big\}-\\ &\sum_{k=1}^{K}\sum_{l=1}^{L}\Omega_{jkl}\Big\{\big[(1-\gamma_{jkl})p_k^{(m-n_{jkl})}+\\ &\gamma_{jkl}\,p_k^{(m-n_{jkl}-1)}\big]+\\ &\hat{t}_{jkl}\big[(1-\gamma_{jkl})\dot{p}_k^{(m-n_{jkl})}+\gamma_{jkl}\,\dot{p}_k^{(m-n_{jkl}-1)}\big]\Big\}.\end{aligned}\tag{17}$$

Special care must be taken in the approximation of the time derivatives present in (17) since this is where the stability problems originate. Mitzner used a second order backward difference formula to approximate the time derivatives. While it is necessary to use a backward difference formula to approximate derivatives at the current time, better approximations can be used for past times. We incorporate smoothing into the derivative approximation by using Q values on either side of the time where the derivative is desired and least-square fitting a quadratic to these values. The derivative is approximated by the derivative of the quadratic at the desired time. This results in a derivative approximation of the form

$$\dot{p}_k^{(m)} \doteq \frac{1}{\Delta t}\sum_{q=-Q}^{Q} C_q\, p_k^{(m-q)}.\tag{18}$$

When $Q=1$, the coefficients are $C_{-1}=1/2$, $C_0=0$, $C_1=-1/2$. This corresponds to a central difference approximation. When $Q=2$, the coefficients are $C_{-2}=2/10$, $C_{-1}=1/10$, $C_0=0$, $C_1=-1/10$, $C_2=-2/10$. The general expression for C_q is

$$C_q=\frac{-3q}{Q(Q+1)(2Q+1)}.\tag{19}$$

Equations (17) and (18) define a recursion relation for the surface pressure. If we group terms by delay in the recursion relation, we obtain a relation of the form

$$p_j^{(m)}=\sum_{k=1}^{K}\sum_{i=0}^{I}A_{ij}^k a_k^{(m-i)}-\sum_{k=1}^{K}\sum_{i=0}^{I}B_{ij}^k p_k^{(m-i)}\tag{20}$$

where

$$I=\mathrm{Int}\left[\frac{\max(r_{jkl})}{c\,\Delta t}\right]+1+Q.$$

Here I corresponds to the largest possible delay over the surface, but for any pair of values j and k there are only a small number of admissible delays. Therefore, most of the coefficients A_{ij}^k and B_{ij}^k in this equation are zero. It is difficult to write general expressions for A_{ij}^k and B_{ij}^k, but it is easy to construct them in a computer program. Clearly there may be several terms in (17) and (18) corresponding to a given delay. The A and B matrices are first initialized to zero. As the terms in (17) and (18) are computed, they can be added into the correct position in the A and B matrices.

Equation (20) can be rearranged to give

$$\begin{aligned}p_j^{(m)}+\sum_{k=1}^{K}B_{0j}^k p_k^{(m)}=&\\ \sum_{k=1}^{K}\sum_{i=0}^{I}A_{ij}^k a_k^{(m-i)}&-\sum_{k=1}^{K}\sum_{i=1}^{I}B_{ij}^k p_k^{(m-i)}\end{aligned}\tag{21}$$

where all the pressure terms at the present time $m\Delta t$ are grouped together on the left-hand side. All the values of p on the right-hand-side are at retarded times. Thus, (21) is a recursion relation for computing the pressure at successive time steps from computed values at previous time steps. We need to solve a system of equations at each time step for the pressures $p_1^{(m)},\ldots,p_K^{(m)}$. However, the matrix for this system of equations only needs to be factored once since it is independent of time. In addition, the matrix is very sparse and heavily diagonally dominant. In fact this matrix is diagonal if the time step Δt is chosen so that

$$\Delta t \le \frac{\min(r_{jkl})_{j\neq k}}{c}.$$

Since most of the elements in the matrices A and B are zero, it is convenient to store the nonzero elements of each matrix in a vector as is often done for sparse matrices. The method we use is sometimes called the skyline method. We first define a single index n corresponding to the pair of indices j,k by

$$n=(k-1)K+j \qquad j,k=1,\ldots,K.\tag{22}$$

We now think of the A and B matrices as being two-dimensional matrices with row index i and column index n. We only store a range of elements in each column corresponding to the admissible delays. In addition, we only sum over the nonzero entries in the recursion relation (21).

As mentioned previously, there are stability problems with the recursion relation (21) when the time step is too small. It may seem odd that stability problems arise for small time steps since, in solving differential equations, the stability problems generally occur for large time steps. However, in solving differential equations the derivative approximations are not used to calculate derivatives, but to calculate values of the underlying function. In fact, the derivative values are given by the differential equations. In the Kirchhoff integral equation we are calculating derivatives at past times in terms of previously calculated pressure values. Since there are errors in these previously computed pressure values that do not go to zero as the time step gets smaller (e.g., spatial discretization errors), small time steps only serve to amplify these errors. The smoothing we use in the derivative approximation reduces the effects of these errors. It should also be mentioned

that it probably doesn't make sense to take arbitrarily small time steps for a given spatial breakup. The time and spatial dependencies are related. If one really needs accuracy at small time steps, the spatial subdivisions need to be chosen appropriately small. It should probably take fewer than ten time steps for an acoustic wave to cross a spatial subdivision. In addition, the spacing between quadrature points should not be too large relative to the time step.

IV. RESULTS

The equations developed in the previous section have been implemented into a computer program that can handle quite general surfaces. The program takes advantage of the sparseness of the matrices involved as well as any geometric symmetries that are present. The surface geometry portion of the program was taken from the frequency-domain program CHIEF [5]. In this section we will show computed results for several sample problems. In order to assess the accuracy of the method we have chosen problems where the solution can also be computed from an analytical expression. The prescribed acceleration waveform for each of these problems is a pulse consisting of one half cycle of a sine wave of angular frequency ω.

The first problem consists of a sphere that is uniformly excited in the radial direction. The time is normalized by the time required for a sound wave to travel across the diameter of the sphere. The pulse width in this normalized time is one. Fig. 1 shows the computed surface pressure at the pole of the sphere. The analytic solution is included for comparison. For this computation there were 8 subdivisions in the θ-direction and 12 subdivisions in the ϕ-direction. The time step in normalized time was 1/30 and the derivatives were approximated using $Q = 1$. The surface pressure is normalized by $\rho ca/\omega$ where a is the amplitude of the acceleration pulse. It can be seen that the agreement is excellent. Fig. 2 shows the error for this same problem. It can be seen that the error at large times is not random, but has a definite pattern. Fig. 3 shows the results from taking an FFT of the error. It can be seen that there are narrow peaks corresponding to the frequencies where the solution to the corresponding Helmholtz integral equation is nonunique. The computed frequencies are shown on the plot with the exact values of the interior resonances shown in parentheses. Fig. 4 shows the computed pressure when the normalized time step is reduced to 1/90. Notice that the solution is unstable at large times. Fig 5. shows the results when Q is increased to 2. Notice that now the solution is stable.

The second problem is similar to the first except that the motion consists of rigid body acceleration in the direction of the polar axis. For this problem we used 24 subdivisions in the θ-direction and 12 subdivisions in the ϕ-direction. The normalized pulse width was 8/10. The pressure is normalized by ρra where r is the radius of the sphere. Fig. 6 shows a comparison between the computed pressure at the pole and the analytic solution. Here again the agreement is excellent. Fig. 7 shows the frequency spectrum of the error. Again there are peaks at the frequencies where the Helmholtz integral equation has a nonunique solution.

The third problem consists of a circular piston at the pole of a sphere. The piston's angular width is 30°. To simplify the analytical solution there is a 30° transition region where the acceleration varies linearly in $\cos\theta$ from one down to zero. Again we are using 24 subdivisions in the θ-direction and a normalized pulse width of 8/10. The normalizations are the same as for the uniformly pulsed sphere. Fig. 8 shows the pressure near the pole (the center of the piston), near the equator, and near the opposite pole. The analytic solution was computed in the frequency domain and numerically transformed to the time domain. Here again the agreement is excellent.

V. CONCLUSIONS

A numerical method has been developed that is stable for small time steps. Comparison with analytic solutions for several test problems has shown that good accuracy can be obtained. In addition it has been found that there are components of the error at frequencies corresponding to resonances of the internal problem (and nonuniqueness of the Helmholtz integral equation solution) that do not decay to zero at large times. These components can, however, be made small.

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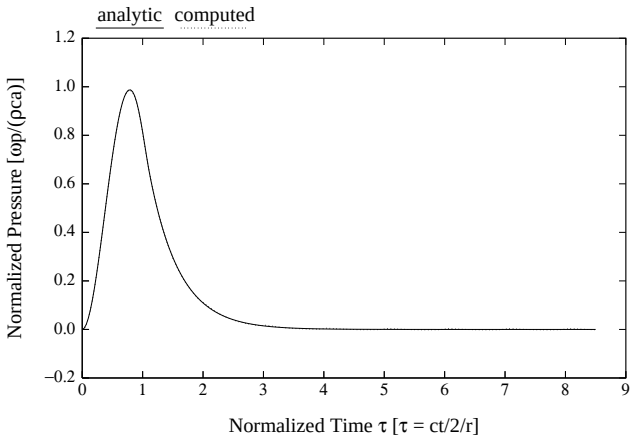

Fig. 1. Surface pressure due to uniformly pulsed sphere

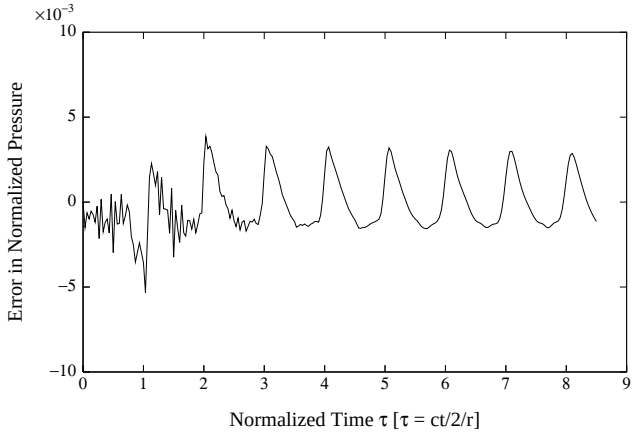

Fig. 2. Error for uniformly pulsed sphere

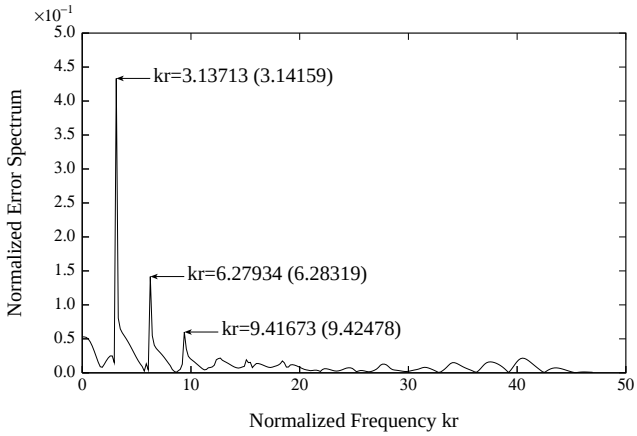

Fig. 3. Error spectrum for uniformly pulsed sphere

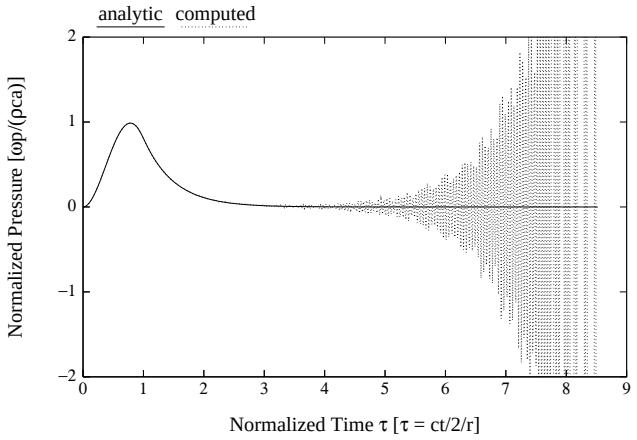

Fig. 4. Unstable behavior of uniformly pulsed sphere for small time steps

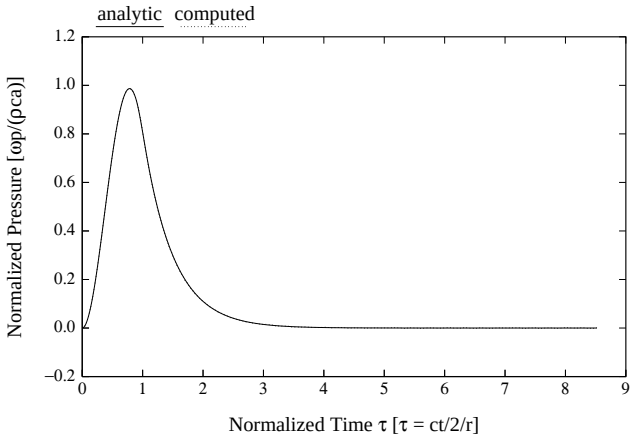

Fig. 5. Stabilization of uniformly pulsed sphere for small time steps

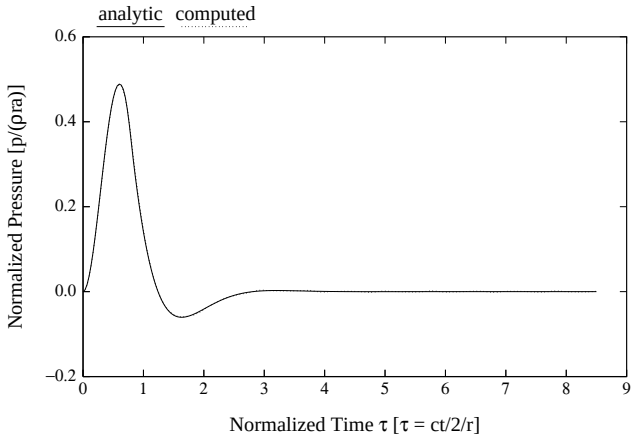

Fig. 6. Surface pressure due to pulsed oscillating sphere

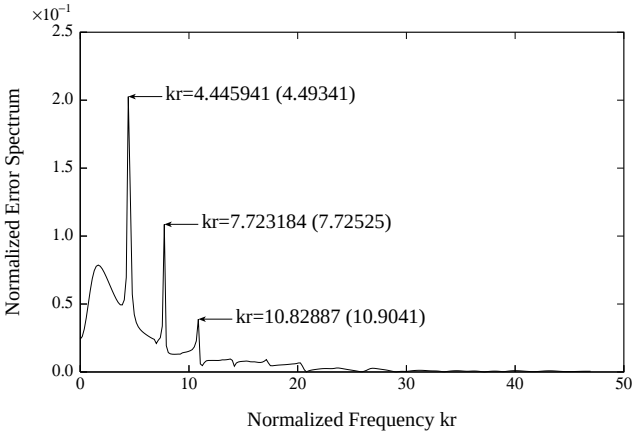

Fig. 7. Error spectrum for pulsed oscillating sphere

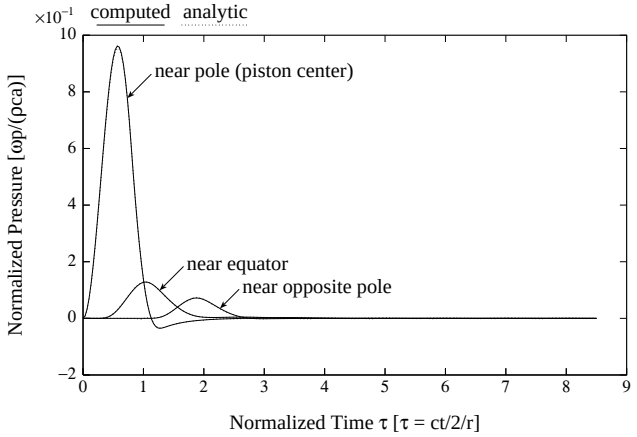

Fig. 8. Surface pressure due to pulsed piston on a sphere