The Statistical Accuracy of an Arctangent Bearing Estimator

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***Abstract*— One of the most widely used underwater acoustic sensors in the world is the DIFAR sonobuoy, which consists of an omni-directional hydrophone and two dipole sensors oriented orthogonally. The directional information provided by the dipoles makes it possible to estimate the bearing to an underwater acoustic source. In this paper, the performance of an arctangent bearing estimator is examined for both sinusoidal and Gaussian signals in isotropic Gaussian noise. It is shown that the estimator is unbiased, and exact formulas for the bearing error (variance) are presented. These formulas allow for the case in which multiple data blocks are averaged before forming the bearing estimate. Several examples based on the numerical evaluation of the bearing variance are presented, and the performance is compared with the Cramér-Rao lower bound.**

I. Introduction

One well-known system for estimating the bearing to an underwater acoustic source consists of two crossed dipoles and an omni-directional hydrophone. This arrangement is used, for example, in the DIFAR sonobuoy. The total bearing error in such a system is a combination of errors caused by many different effects. Although many of the error components can be reduced through improvements in system construction, there is an unavoidable bearing error induced by the ambient ocean noise that is present in the acoustic signals. This noise places an ultimate limitation on the achievable bearing accuracy and in a well-designed system will be the dominant source of error. The focus of the paper is this noise-induced error.

Some analysis has been done previously on the variance of the bearing error for a crossed-dipole sensor when the bearing estimator uses an arctangent function [1], [2]. In these studies the signal and noise were assumed to have Gaussian statistics, and the noise was taken to be spatially isotropic. Davies noted the difficulty of deriving an exact expression for the bearing error, and used an approximate approach based on the central limit theorem [2]. His approximate results are valid only when a large number of data blocks are integrated before forming the bearing estimate; for example, by using computer simulations he found that his analytical approximation is good only to an accuracy of 20% when 40 blocks are averaged.

In this paper, the performance of the arctan bearing estimator is examined for both sinusoidal and Gaussian signals in isotropic Gaussian noise. It is shown that the estimator is unbiased, and exact formulas for the bearing error (variance) are presented. These formulas can handle the case in which an arbitrary number of data blocks are averaged before forming the bearing estimate. Several illustrative examples based on the numerical evaluation of the bearing variance are presented. Performance comparisons are made between the two signal types (sinusoid and Gaussian), and the results are also compared to the Cramér-Rao lower bound.

Although the present paper considers only a single DIFAR sensor, a vertical line array [3] has been built from such sensors. More generally, particle-velocity sensors of various configurations have been incorporated into arrays; for example, DRDC Atlantic has built a towed array containing omni-directional hydrophones and dipoles that point in the transverse direction. See [4] and the references therein for details on how such arrays are processed.

II. The Arctangent Bearing Estimator

A. Pre-processing of the Data

As a first step in the discussion of the crossed-dipole sensor, let us consider the analytic form of the waveforms received by the three sensor channels, which are arranged as shown in Fig. 1. The subscripts o, c, and s will be used to designate the omni, cosine, and sine channels. When the source is at bearing β, the sensors output the waveforms

$$\begin{aligned} x_o(t) &= y(t) + n_o(t) \\ x_c(t) &= y(t)\cos\beta + n_c(t) \\ x_s(t) &= y(t)\sin\beta + n_s(t), \end{aligned} \tag{1}$$

where $y(t)$ is the signal component, and $n_o(t)$, $n_c(t)$, and $n_s(t)$ are noise components. It has been assumed that the propagating signal arrives in the horizontal plane. As noted in the introduction, the noise components arise from the ambient ocean noise.

The input data to the bearing estimator will be the Fourier coefficients of the waveforms in (1). Since the signals are observed for a finite time, the Fourier transform of an arbitrary signal $f(t)$ will be defined by

$$F(\omega) = \frac{1}{T}\int_0^T f(t)e^{-i\omega t}dt, \tag{2}$$

where T is the observation time, or block length. (Analog signals and processing will be assumed throughout the analysis

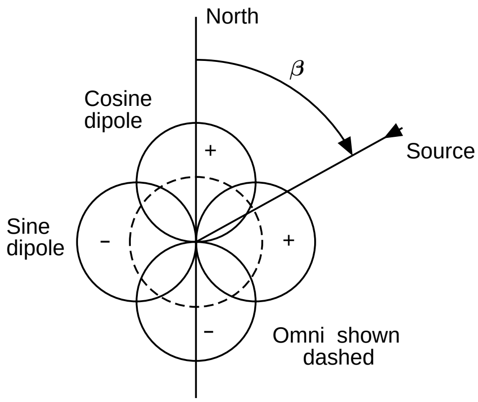

Fig. 1. Geometric arrangement of the three sensors (top view).

as a matter of convenience; the results would be the same for the equivalent digital processing.)

After Fourier transformation, (1) can be written in the vector form $\mathbf{x} = \mathbf{y} + \mathbf{n}$ with

$$\mathbf{x} = \begin{pmatrix} X_o \\ X_c \\ X_s \end{pmatrix}, \quad \mathbf{y} = Y \begin{pmatrix} 1 \\ \cos\beta \\ \sin\beta \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} N_o \\ N_c \\ N_s \end{pmatrix}. \qquad (3)$$

In these equations, the frequency argument ω has been omitted from the Fourier coefficients; i.e., $X_o = X_o(\omega)$, etc.

Usually, multiple data blocks, or “snapshots”, are processed. That is, multiple data vectors $\mathbf{x}_m$ are produced during the pre-processing stage before the bearing is estimated. Here the subscript m indicates the block number; it will be assumed that a total of M blocks are available, so that $1 \leq m \leq M$.

B. The Bearing Estimator

We now consider a specific bearing estimator called the arctan estimator, which is implemented as follows: given the measured data vectors $\mathbf{x}_m$, we first compute

$$\hat{c} \equiv \mathrm{Re} \sum_{m=1}^{M} X_{c,m} X_{o,m}^{*} \qquad (4)$$

$$\hat{s} \equiv \mathrm{Re} \sum_{m=1}^{M} X_{s,m} X_{o,m}^{*} \qquad (5)$$

and then form a bearing estimate using the equation

$$\hat{\beta} = \arctan(\hat{s}/\hat{c}). \qquad (6)$$

Although the arctan bearing estimator is usually written as in (6), the ratio $\hat{s}/\hat{c}$ is in fact insufficient to form a bearing estimate, as it is necessary to know the signs of $\hat{s}$ and $\hat{c}$ in order to determine the correct quadrant. More precisely, $\hat{\beta}$ is determined from the polar transformation

$$\begin{aligned} \hat{c} &= \hat{r}\cos\hat{\beta} \\ \hat{s} &= \hat{r}\sin\hat{\beta}, \end{aligned}$$

where $\hat{r}$ is positive and $\hat{\beta}$ is chosen to lie in an interval of length 2π.

III. Signal and Noise Models

In order to analyze the statistical performance of the arctan bearing estimator, models are required for the signal and the noise. The signal will be modeled in two different ways: as a sinusoid, and as a Gaussian random process. The latter model should more accurately represent a signal received at long range in the ocean than does the deterministic sinusoid; a comparison of the two cases is therefore of interest. The models are first described for a single data block, and later extended to multiple blocks. In this way, the basic argument is not obscured by unnecessarily complicated notation.

A. Noise Statistics

The noise will be assumed to have Gaussian statistics, and therefore is completely described by its mean and covariance. The mean is taken to be zero; i.e., $E\{\mathbf{n}\} = \mathbf{0}$. It remains to specify the covariance matrix, given by $\mathbf{K}_n \equiv E\{\mathbf{n}\mathbf{n}^{\dagger}\}$, where the dagger denotes the conjugate transpose. The only noise distributions considered will be 2D- and 3D-isotropic noise, for which the covariance matrix assumes the simple form

$$\mathbf{K}_n = \sigma_n^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \qquad (7)$$

where $\sigma_n^2 \equiv E\{|N_o|^2\}$ is the noise power in the Fourier bin of the omni channel and γ is the noise gain of either dipole. The value of $\gamma = 1/2$ or 1/3 for 2D-isotropic or 3D-isotropic noise respectively [2]. The off-diagonal terms of $\mathbf{K}_n$ are all zero for isotropic noise.

B. Signal Model I: Sinusoid

For the first signal model, the signal has the form of a sinusoid

$$y(t) = 2A\cos(\nu t + \psi), \qquad (8)$$

where $2A$ is the amplitude, ν the frequency, and ψ an arbitrary phase. The amplitude has been written as $2A$ to simplify the subsequent equations.

For simplicity, it will be assumed that the front-end Fourier transforms are taken exactly at the frequency of the sinusoidal signal; i.e., $\omega = \nu$. From (2) and (8), it is found that

$$Y = A\left\{ e^{i\psi} + e^{-i(\omega T + \psi)} \frac{\sin(\omega T)}{\omega T} \right\}.$$

If it is assumed that $\omega T \gg 1$ (i.e., that there are many cycles of the sinusoidal signal in the observation time T), then to a high degree of approximation

$$Y \cong A e^{i\psi}. \qquad (9)$$

In other words, it will be assumed that the second term falls into the sidelobe region of the rectangular window, and can therefore be neglected. The signal component in (3) thus has the form

$$\mathbf{y} = A e^{i\psi} \begin{pmatrix} 1 \\ \cos\beta \\ \sin\beta \end{pmatrix}. \qquad (10)$$

We shall denote by ρ the signal-to-noise ratio (SNR) in the Fourier bin of the omni channel. The signal component in the bin is given by (9), and we have

$$\rho = \frac{|Y|^2}{E\{|N_o|^2\}} = \frac{A^2}{\sigma_n^2}. \tag{11}$$

The SNR in the bin can be related to the input quantities as follows. First, the average power of the signal in (8) is $P_s = \frac{1}{2}(2A)^2 = 2A^2$. Also, if the noise $n_o(t)$ recorded by the omni sensor is white with two-sided power spectral density $\eta/2$, then it can be shown from (2) that $\sigma_n^2 = \eta/2T$. Hence the SNR as defined above is equal to

$$\rho = \frac{P_s T}{\eta} = \frac{P_s}{\eta W},$$

where $W = 1/T$ is the nominal bandwidth of the Fourier analysis cell. In words, the SNR is given by the ratio of the average signal power to the average noise power entering into the bin. If the time functions are weighted before the Fourier transform is performed, the SNR is reduced by the equivalent noise bandwidth of the weighting function.

C. Signal Model II: Gaussian Signal

For the second signal model, $y(t)$ is taken to be a stationary Gaussian process. The Fourier coefficient Y of the signal is then also Gaussian distributed, and it is assumed to have zero mean, $E\{Y\} = 0$. The variance is denoted by $\sigma_y^2 \equiv E\{|Y|^2\}$, and in analogy with (11) the SNR is given by

$$\rho = \frac{E\{|Y|^2\}}{E\{|N_o|^2\}} = \frac{\sigma_y^2}{\sigma_n^2}. \tag{12}$$

It follows from the above that the measured data vector $\mathbf{x}$ has a complex Gaussian distribution with zero mean and covariance matrix

$$\mathbf{K}_x = \sigma_n^2 \begin{pmatrix} \rho+1 & \rho\cos\beta & \rho\sin\beta \\ \rho\cos\beta & \rho\cos^2\beta+\gamma & \rho\cos\beta\sin\beta \\ \rho\sin\beta & \rho\cos\beta\sin\beta & \rho\sin^2\beta+\gamma \end{pmatrix}, \tag{13}$$

where it has been assumed that the signal and noise are statistically independent.

D. Multiple Data Blocks

To extend the model to multiple blocks, the measured data is represented by M data vectors $\mathbf{x}_m$, each having the same statistical distribution as a single block $\mathbf{x}$. It will be assumed that the vectors are statistically independent, an assumption that at the very least implies that the M front-end Fourier transforms operate on disjoint time intervals (i.e., there is no overlap of the input time series).

IV. Analytical Results

Exact formulas have been derived for the variance of the arctan bearing estimator $\hat{\beta}$, for both the sinusoidal and the Gaussian signal. Space limitations preclude presenting these long derivations, and for this reason only the final formulas will be given.

A. Bias and Variance

The arctan bearing estimator $\hat{\beta}$ is unbiased for either signal model. It should be noted that this statement refers to $\hat{\beta}$ taken modulo 2π, as the unbiased nature of the estimator is obscured when the circle is broken into a linear segment. (See §5.33 of [5] for a discussion of distributions on the circle. An illuminating exchange on the definition of bias for such distributions can be found in [6].)

The above-stated fact can be formulated analytically as follows. First, let $f(\hat{\beta}|\beta)$ denote the probability density function (pdf) of $\hat{\beta}$ when the true bearing angle is β. Next, define a complex variable ζ by

$$\zeta = \int \exp(i\hat{\beta})\, f(\hat{\beta}|\beta)\, d\hat{\beta},$$

where the range of integration is any interval of length 2π. Then it can be shown that $\beta = \arg(\zeta) \bmod 2\pi$ for both the sinusoidal and Gaussian signals, demonstrating the unbiased nature of the estimator in the two cases.

It is not as easy to formulate the variance on the circle. The approach taken here is to break the circle into a linear interval, centering it on the true value β in order to minimize the effect of introducing truncation at the interval ends. Then the variance is defined as

$$\begin{aligned} \sigma_{\hat{\beta}}^2 &= \int_{\beta-\pi}^{\beta+\pi} (\hat{\beta}-\beta)^2 f(\hat{\beta}|\beta)\, d\hat{\beta} \\ &= \int_{-\pi}^{\pi} \theta^2 f(\theta|0)\, d\theta. \end{aligned} \tag{14}$$

The last step follows from the fact that $f(\hat{\beta}|\beta) = f(\hat{\beta}-\beta|0)$, as can be shown for both the sinusoidal and Gaussian signal models. When the SNR is sufficiently high that $f(\theta|0)$ is peaked sharply around the origin and drops off near the interval ends, (14) provides a physically meaningful measure of variance. This will be the case in all the numerical examples shown later.

B. Signal Model I: Sinusoid

An exact formula for the variance when the signal is a sinusoid is given by [7]

$$\begin{aligned} \sigma_{\hat{\beta}}^2 &= \frac{(\rho')^M e^{-\rho'}}{\sqrt{\pi}\,\Gamma(M-\frac{1}{2})} \int_{-\pi}^{\pi} K\Big(\sqrt{\tfrac{\rho'}{\gamma}}\cos\theta\Big) \sin^{2M-2}\theta \\ &\quad \times \int_0^\infty \exp\big[-\rho'(t^2 - 2t\cos\theta)\big] t^{2M-1}\, dt\, d\theta, \end{aligned} \tag{15}$$

where $\rho' \equiv M\rho$ and $K(\cdot)$ is an auxiliary function defined by the double integral

$$K(x) = \frac{1}{\pi}\int_{-\pi}^{\pi}\int_0^\infty \theta^2 \exp\big[-(x^2+t^2-2xt\cos\theta)\big]\, t\, dt\, d\theta. \tag{16}$$

Equations (15) and (16) appear formidable, and it is worthwhile to pause and consider how they can be evaluated numerically. First, the author has written a computer routine to evaluate $K(\cdot)$ based on asymptotic expansions and Chebyshev polynomial approximations. Second, the inner integral in (15),

$$\int_0^\infty \exp\left[-\rho'(t^2-2t\cos\theta)\right]t^{2M-1}\,dt,$$

can be readily evaluated for arbitrary M using a recursion relation. Thus, only the outer integral in (15) need be done by numerical quadrature; high speed and accuracy can consequently be attained in the numerical evaluation of the bearing variance.

C. Signal Model II: Gaussian Signal

For convenience in what follows, we denote the six distinct elements in the matrix $\mathbf{K}_x$ of (13) according to the scheme

$$\mathbf{K}_x = \begin{pmatrix} \alpha_1 & \alpha_4 & \alpha_5 \\ \alpha_4 & \alpha_2 & \alpha_6 \\ \alpha_5 & \alpha_6 & \alpha_3 \end{pmatrix}. \tag{17}$$

Although in this specific case all the entries in the matrix are real-valued, we allow for the possibility that the off-diagonal terms α_4 through α_6 are complex-valued, as could occur if the ambient noise were anisotropic. Thus the result presented here may be applied more generally.

We next introduce into the analysis a vector $\mathbf{d}$ and matrix $\mathbf{D}$, defining them as

$$\mathbf{d} = \alpha_1^{-1}\mathrm{Re}\begin{pmatrix} \alpha_4 \\ \alpha_5 \end{pmatrix}$$

and

$$\mathbf{D} = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{12} & \xi_{22} \end{pmatrix},$$

where

$$\begin{aligned} \xi_{11} &= \alpha_1^{-2}(\alpha_1\alpha_2 - |\alpha_4|^2) \\ \xi_{22} &= \alpha_1^{-2}(\alpha_1\alpha_3 - |\alpha_5|^2) \\ \xi_{12} &= \alpha_1^{-2}\mathrm{Re}\{\alpha_1\alpha_6 - \alpha_4\alpha_5^*\}. \end{aligned}$$

The ξ factors have been scaled such as to make them dimensionless.

It can be shown that the pdf $f(\hat\beta|\beta)$ of the bearing estimate $\hat\beta$ is given by

$$f(\hat\beta|\beta) = \frac{M}{\pi|\mathbf{D}|^{1/2}}\int_0^\infty \frac{t\,dt}{Q(t,\hat\beta)^{M+1}}, \tag{18}$$

where $Q(t,\hat\beta)$ is a quadratic function of the integration variable t defined by

$$Q(t,\hat\beta) \equiv (t\,\mathbf{u}_{\hat\beta} - \mathbf{d})^T\mathbf{D}^{-1}(t\,\mathbf{u}_{\hat\beta} - \mathbf{d}) + 1$$

and

$$\mathbf{u}_{\hat\beta} \equiv \begin{pmatrix} \cos\hat\beta \\ \sin\hat\beta \end{pmatrix}.$$

The pdf in (18) can be expressed in closed form for $M = 1$, but in general it is necessary to perform the integration appearing there numerically. An additional numerical integration must be performed to compute the variance via (14).

V. The Cramér-Rao Lower Bound

It is instructive to compare the variance of the arctangent bearing estimator with the Cramér-Rao lower bound (CRLB), which provides a theoretical limit on the variance achievable by any unbiased estimator. The CRLB is derived here for the signal-and-noise models of Sec. III and is used later in the numerical examples. Once again, we give the main derivation for the case of a single data block, and then remark on how the results are extended to the case of multiple data blocks.

A. Calculating the CRLB

The observed data is the vector $\mathbf{x}$, the pdf of which will be denoted by $f(\mathbf{x}|\mathbf{p})$, where $\mathbf{p}$ is a vector of parameters on which the pdf depends. In the problem at hand, the parameter of interest is the bearing β; however, all relevant parameters must enter into $\mathbf{p}$. To compute the CRLB we first form the Fisher information matrix $\mathbf{J}$ with (i,j)-th element

$$J_{ij} \equiv -E\left\{\frac{\partial^2}{\partial p_i\partial p_j}\log f(\mathbf{x}|\mathbf{p})\right\}.$$

Here the expectation $E\{\cdot\}$ is taken with respect to $f(\mathbf{x}|\mathbf{p})$. Letting J^{ij} denote the (i,j)-th element of the inverse $\mathbf{J}^{-1}$ of the Fisher matrix, the CRLB on the variance of any unbiased estimator $\hat p_i$ of the parameter p_i is given by $\sigma^2_{\hat p_i} \geq J^{ii}$.

For the two signal-and-noise models of Sec. III, the data vector $\mathbf{x}$ is distributed according to the complex Gaussian law

$$f(\mathbf{x}|\mathbf{p}) = \frac{1}{\pi^3\det\mathbf{K}}\exp[-(\mathbf{x}-\mathbf{m})^\dagger\mathbf{K}^{-1}(\mathbf{x}-\mathbf{m})], \tag{19}$$

where the mean vector $\mathbf{m}$ and the covariance matrix $\mathbf{K}$ depend on $\mathbf{p}$. It may be shown that, for this specific distribution, the elements of the Fisher matrix are given by [8]

$$J_{ij} = \mathrm{tr}\left\{\mathbf{K}^{-1}\frac{\partial\mathbf{K}}{\partial p_i}\mathbf{K}^{-1}\frac{\partial\mathbf{K}}{\partial p_j}\right\} + 2\,\mathrm{Re}\left\{\frac{\partial\mathbf{m}^\dagger}{\partial p_i}\mathbf{K}^{-1}\frac{\partial\mathbf{m}}{\partial p_j}\right\},$$

where $\mathrm{tr}\{\cdot\}$ denotes the matrix trace operation. It is now a matter of "turning the crank" to calculate the CRLB for the two signal-and-noise models.

B. Signal Model I: Sinusoid

In this case, $\mathbf{m}$ and $\mathbf{K}$ in (19) are respectively equal to the signal vector $\mathbf{y}$ in (10) and the noise covariance matrix $\mathbf{K}_n$ in (7). The parameter vector $\mathbf{p}$ has five entries, namely $\mathbf{p} = (\beta, A, \psi, \sigma_n, \gamma)^T$. After performing the requisite calculations, it is found that the CRLB on any unbiased bearing estimator $\hat\beta$ is given by

$$\sigma^2_{\hat\beta} \geq \frac{\gamma}{2\rho}. \tag{20}$$

As expected, the bound decreases as ρ increases; i.e., as the SNR improves, more accurate bearing estimates are theoretically possible.

C. Signal Model II: Gaussian Signal

In this case, $\mathbf{m} = \mathbf{0}$ and $\mathbf{K} = \mathbf{K}_x$, where $\mathbf{K}_x$ is given in (13). The parameter vector now has four entries, $\mathbf{p} = (\beta, \rho, \sigma_n, \gamma)^T$. Working through the calculations once again, we find that the CRLB for the Gaussian signal is given by

$$\sigma_{\hat{\beta}}^2 \geq \frac{\gamma(\gamma + \rho + \gamma\rho)}{2(1+\gamma)\rho^2}. \tag{21}$$

With appropriate changes in notation, this expression for the CRLB agrees with that derived in [9].

For values of ρ greater than a few dB, the right-hand side of (21) is accurately approximated by the result in (20). That is, in the SNR region of interest, the CRLB is approximately the same for both sinusoidal and Gaussian signals.

D. Multiple Data Blocks

The data are now the M vectors $\mathbf{x}_m$, which are independent and identically distributed. It is readily seen that the entries in the Fisher matrix are simply multiplied by a factor M, so that the bounds are correspondingly decreased by a factor M. For example, the bound in (20) becomes

$$\sigma_{\hat{\beta}}^2 \geq \frac{\gamma}{2M\rho} \tag{22}$$

for the multiple snapshot model.

VI. Numerical Examples

Several numerical examples are now given to illustrate the application of the foregoing formulas. In all examples we shall use the value $\gamma = 1/3$, corresponding to the physical situation of 3D-isotropic noise.

A. Signal Model I: Sinusoid

We first consider the case when the signal is a sinusoid. In Fig. 2, the standard deviation $\sigma_{\hat{\beta}}$ of the arctan bearing estimator is plotted as a function of the signal-to-noise ratio ρ for different values of M. The curves were computed using (15), and the circles that appear on the curves for $M = 2$ and 8 are the results of Monte Carlo simulations [7]. Clearly there is excellent agreement between the theoretical and the simulated results.

As one would expect, the bearing error decreases as one increases the number of data blocks M used in forming the bearing estimate, and it is seen from Fig. 2 that using multiple data blocks provides the most benefit when trying to improve bearing accuracy at low SNR.

Fig. 3 compares the standard deviation of the arctan bearing estimator (solid lines) to the Cramér-Rao lower bound in (22) (dashed lines). It is seen that the standard deviation is close to the theoretical bound when the SNR is high, but pulls away as the SNR drops. Such behavior is often observed in non-linear estimators, and is referred to as a threshold effect. The threshold extension (i.e., lowering of the threshold) obtained by using multiple data blocks in the bearing estimator is evident in Fig. 3.

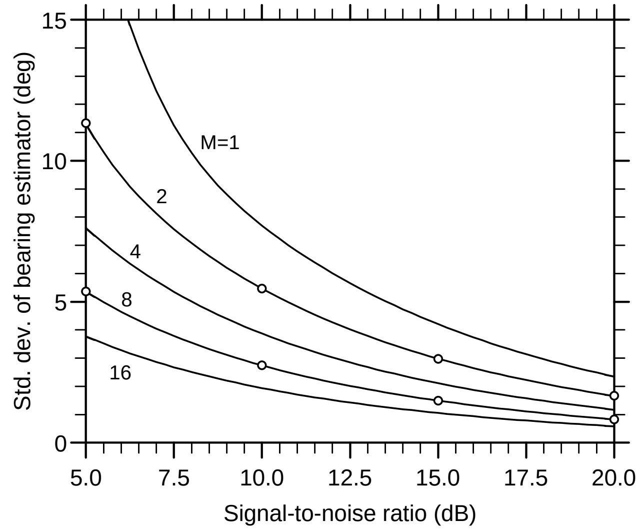

Fig. 2. Standard deviation of the arctan estimator versus the signal-to-noise ratio for a sinusoidal signal. Circles indicate simulated data points.

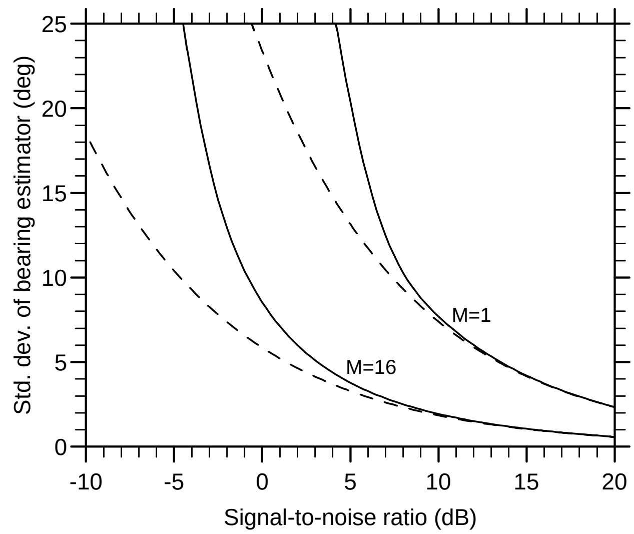

Fig. 3. Standard deviation of the arctan estimator (solid lines) compared with the Cramér-Rao lower bound (dashed lines).

B. Signal Model II: Gaussian Signal

We next turn to the case of the Gaussian signal, plotting in Fig. 4 the standard deviation of the arctan estimator as a function of SNR. The curves were computed numerically using formulas (14) and (18), and the circles indicate the results from Monte Carlo simulations. As before, the agreement between the theory and simulation is excellent.

To facilitate comparison of the results for the Gaussian signal with those for the sinusoidal signal, the axes in Fig. 4 were chosen to be the same as in Fig. 2. Comparison of the two figures shows that the arctan estimator provides roughly the same level of performance for both sinusoidal and Gaussian

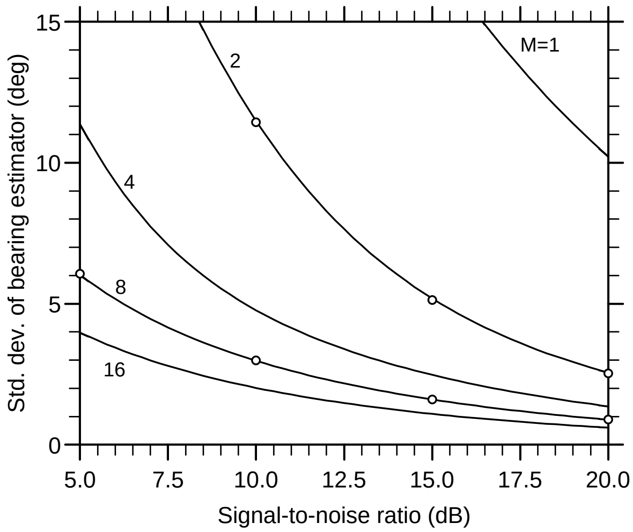

Fig. 4. Standard deviation of the arctan estimator versus the signal-to-noise ratio for a Gaussian signal. Circles indicate simulated data points.

signals when many blocks are averaged (i.e., when M is large); however, the bearing error is much larger for the Gaussian signal when only a few blocks are used. The reader is reminded that the CRLB for the bearing variance is approximately the same for both sinusoidal and Gaussian signals in the SNR range of interest.

The author posits the following explanation for the different behavior of the estimator for the two signal types. On the one hand, since the sinusoidal signal is deterministic, it appears with the same energy in every data snapshot, and there is no risk of missing the signal by looking at only a single data block. On the other hand, the Gaussian signal is a stochastic process that is characterized by its *mean* energy per snapshot [compare the definitions of the SNR in (11) and (12)]. Since the energy of the Gaussian signal fluctuates from snapshot to snapshot, there is a possibility of obtaining a single data block in which the signal energy is very low. Only by averaging a sufficiently large number of blocks is the statistical variability reduced and a more stable estimate provided.

C. Discussion

It has been seen that the variance of the arctan bearing estimator approaches the CRLB only under certain conditions. One drawback of the CRLB is that it provides no insight into how to construct estimators, and there is no *a priori* way to tell if an estimator that achieves the bound even exists. Maximum likelihood (ML) estimation requires the solution of non-linear equations for the problem considered here, and the simple arctan estimator is used in practical applications. The author suspects that there is no estimator that will substantially outperform the arctan estimator in those regions where it exceeds the CRLB by a large margin.

One point to be borne in mind is the way in which multiple data blocks are used in the processing: they are averaged before the bearing estimate is made, as indicated in (4) and (5). There is an alternative strategy, that of forming a bearing estimate for each of the M data blocks and then averaging the M estimates. Davies [2] calls this latter method "post-bearing averaging", and notes that in general it is inferior to the method that averages the data and then forms the bearing estimate.

VII. Conclusion

Using a sensor unit formed from an omni-directional hydrophone and two crossed dipoles, it possible to estimate the bearing to an acoustic source. In this paper, a specific type of bearing estimator based on the arctangent function was analyzed. Formulas for the variance of the estimator were presented for two signal-and-noise models, the first assuming a sinusoidal signal and the second a Gaussian signal. In both cases the ambient ocean noise was modeled as Gaussian noise with an isotropic spatial distribution.

Numerical examples showed that the arctangent estimator provides more accurate bearings for a sinusoidal signal than for a Gaussian signal at the same signal-to-noise ratio. However, the difference in performance is reduced by averaging the data from a sufficiently large number of snapshots, and approximately equivalent performance is obtained for an SNR greater than 5 dB when the number of snapshots is 8 or more.

Although the ambient noise was modeled as isotropic in this paper, the formulas for the case when the signal is Gaussian allow for anisotropic noise and hence can be used for investigations of more general scenarios (for example, to examine the estimator bias when the noise is anisotropic).

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