A Normal-mode Expression for the Sensitivity of Acoustic Fields to Three-dimensional Refractive Index Perturbations in a Constant-depth Waveguide, using an Analytic Adjoint Approach

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Abstract-**Adjoint or "co-state" methods have recently been applied to the ocean acoustic inverse problem using both a 2-D range-dependent PE approach [P.Hursky, M. Porter, M. Porter, B. Cornuelle, W.S. Hodgkiss, W.A. Kuperman, J. Acous. Soc. Am. 112(5) p. 2402] and a range-independent normal mode formulation[A. Thode, J. Acous. Soc. Am. 112(5), p. 2221]. Here the normal-mode formulation is extended to yield expressions for the derivative of an acoustic field with respect to an arbitrary three-dimensional refractive index perturbation in the water or bottom, for the restricted case of a constant-depth waveguide. These results may be useful as both an independent check for 3-D propagation codes [e.g. K. B. Smith, J. Acous. Soc. Am. 106(6), 3231-3239] and for ocean acoustic tomography sensitivity studies. For example, inversions for sound speed perturbations in an ocean waveguide are generally conducted along a vertical plane connecting source and receiver, thus neglecting the potential effects of horizontal refraction by "out-of-plane" perturbations. A simple sensitivity analysis of the effect of out-of-plane perturbations on a propagating modal field is presented.**

I. INTRODUCTION

This paper summarizes some formulas that have been developed to compute the derivative of an acoustic field with respect to an arbitrary three-dimensional refractive index perturbation in a constant-depth waveguide, where the perturbation can lie in either the water column or the ocean bottom. These “environmental derivatives” are expressed in terms of the waveguide normal modes and horizontal wavenumbers. These formulas have been used to compute the environmental derivatives of both 3-D plane-wave and compact cylindrical perturbations in a waveguide. The plane-wave perturbation analysis shown here provides one demonstration of how perturbation components outside the vertical range-depth plane connecting source and receiver can affect the environmental derivative. In other words, some form of horizontal refraction occurs.

II. THEORY

A. Adjoint formula for environmental derivative

A monochromatic acoustic pressure field $g(\bar{r}_i,\bar{r}_s,\omega)$ generated at location r_s propagates through an unknown environment, and is measured at location r_i. The instrument positions are expressed in terms of cylindrical coordinates. The receiver lies on the origin, and the source along the x-axis (see Figure 1). The propagation of the acoustic field is governed by the density-dependent linear wave equation:

$$\rho(\bar{r})\nabla\bullet\left(\frac{1}{\rho(\bar{r})}\nabla g(\bar{r},\bar{r}_s,\omega)\right)+k_{ref}^2\eta(\bar{r})g(\bar{r},\bar{r}_s,\omega)=\delta(\bar{r}-\bar{r}_s,\omega) \tag{2.1}$$

where k_{ref} is a spatially-invariant medium wavenumber. The quantity $\eta(\bar{r},\omega)\equiv k(\vec{r})^2/k_{ref}^2=c_{ref}^2/c(\vec{r})^2$ describes a spatially-dependent sound speed refractive index perturbation that defines the propagation environment.

Let the refractive index perturbation be written as the sum of a range-independent (but potentially depth dependent) “background” perturbation η_0, and a three-dimensional perturbation weighted by a scalar parameter ε:

$$\eta(\bar{r})\equiv\eta_0(\bar{r})+\varepsilon\eta_\varepsilon \tag{2.2}$$

Using standard adjoint techniques developed in the geophysical [1;2] and acoustic[3] literature, the derivative of the pressure field with respect to ε can be shown to be

$$\frac{\partial g(\bar{r}_i,\bar{r}_s)}{\partial\varepsilon}=k_{ref}^2\iiint_V\left[\eta_\varepsilon(\bar{r})\frac{g(\bar{r},\bar{r}_i)}{\rho(\bar{r})}g(\bar{r},\bar{r}_s)\right]\mathbf{d^3r} \tag{2.3}$$

Equation (2.3) requires a spatial correlation of two Greens functions, weighted by the value of the three-dimensional perturbation at each point in space.

B. Basis functions

It can be shown that the set of orthogonal basis functions

$$\eta_{pqv}(\bar{r},z)=a_{pqv}\eta_q(z)e^{-iv\phi}J_v\left(k_p|r|\right) \tag{2.4}$$

can reconstruct any arbitrary 3-D perturbation, for appropriately chosen coefficients a_{pqv}. For example, a horizontal 2-D plane wave with spatial wavenumber k_{pw} propagating along azimuth φ_p with no depth-dependence can be written

$$e^{-i\bar{k}_{pw}\bullet\bar{r}}=e^{-ik_{pw}r\cos(\phi-\phi_p)}=\sum_{v=0}^{\infty}\beta_v(-i)^v\cos\left[v(\phi-\phi_p)\right]J_v\left(k_{pw}r\right) \tag{2.5}$$

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where β_v=1 when v is 0, and is 2 otherwise.

C. Environmental derivative of basis function

In a range-independent waveguide the Greens function can be written as a sum of normal modes U_f with associated horizontal wavenumbers k_{rf}:

$$g(\bar{r},\bar{r}_i) = \frac{i}{4\rho(z_i)}\sum_f U_f(z)U_f(z_i)H_0^{(1)}\left(k_{rf}|\bar{r}|\right) \qquad (2.6)$$

Using methods similar to those used in computing ambient noise in range-dependent environments[4], it can be shown that combining (2.3), (2.4), and (2.6) yields

$$\frac{\partial\psi(\bar{r}_i,\bar{r}_s)}{\partial a_{pqv}} = \left[\frac{-\pi}{8\rho(z_s)\rho(z_i)}\sum_{f,g} Z_{qfg}U_f(z_i)U_g(z_s)\sum_{v=-\infty}^{\infty} R_{pvfg}(r_s)e^{-iv\phi_s}\right] \qquad (2.7)$$

where

$$Z_{qfg} \equiv k_{ref}^2\int_0^\infty \frac{\eta_q(z')}{\rho_0(z')}U_f(z')U_g(z')dz', \qquad (2.8)$$

is a symmetric square matrix, and

$$R_{pvfg}(r_s) = \frac{r_s}{\pi}\int_0^\pi \frac{\cos[v\phi]d\phi}{k_{rpg}^2 - k_{rf}^2}\left\{\begin{array}{l} k_{rpg}H_0^{(1)}(k_{rf}r_s)\left[H_v^{(1)}(k_{rg}r_s)J_1(k_{rpg}r_s) - H_1^{(1)}(k_{rpg}r_s)J_v(k_{rg}r_s)\right] \\ -k_{rf}H_1^{(1)}(k_{rf}r_s)\left[H_v^{(1)}(k_{rg}r_s)J_0(k_{rpg}r_s) - H_0^{(1)}(k_{rpg}r_s)J_v(k_{rg}r_s)\right] \\ -2iH_v^{(1)}(k_{rg}r_s)/\pi \end{array}\right\} \qquad (2.9)$$

$$k_{rpg}^2 = k_{rg}^2 + k_p^2 - 2k_{rg}k_p\cos\phi$$

The last equation involves a straightforward numerical integration over a well-defined azimuthal region. Equations (2.7)-(2.9) can be pre-computed for a set of basis functions, which can then be combined to generate the derivative with respect to an arbitrary perturbation.

III. NUMERICAL EXAMPLE

C. Effect of transverse 3-D plane-wave perturbation on environmental derivatives

Many ocean acoustic tomography studies restrict the modeled acoustic field to a vertical plane containing both source and receiver, neglecting potential out-of-plane (i.e. horizontal refraction) effects from a perturbation. The conditions under which this assumption is valid can be easily checked using the formulas developed in the previous section.

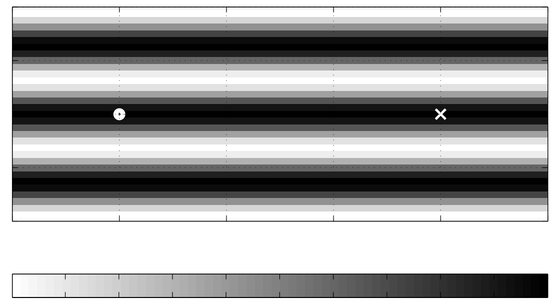

Fig. 1. Top view of cosine plane-wave perturbation relative to source('x') and receiver('o') location. Note that the perturbation value along a vertical slice connecting source and receiver will be the same regardless of the perturbation wavelength chosen. The perturbation wavelength illustrated here is 639 m.

For example, Figure 1 illustrates a situation where a plane-wave perturbation is propagating perpendicular to the vertical plane containing an acoustic source and receiver. Since the perturbation is a cosine function, the perturbation value within the vertical plane is constant, regardless of the perturbation wavelength. If out-of-plane effects could be neglected, then the environmental pressure derivative should remain unchanged by changes in the perturbation wavelength.

Figure 2 illustrates the results of applying (2.7)-(2.9) to the situation illustrated in Figure 1, for a large number of perturbation wavelengths embedded in a Pekeris waveguide bottom. The source frequency is 20 Hz, the water depth 100 m, and the bottom parameters chosen to be similar to sand. For small wavenumbers (large wavelengths), the effect of out-of-plane perturbations appears small. However once the perturbation wavelength becomes smaller than 300 m the environmental pressure derivative changes rapidly, even though the in-plane perturbation value never changes.

This simple illustration provides one example of how these normal mode formulas can be used for sensitivity studies, and perhaps could even be used for local inversion techniques for small-amplitude perturbations, such as would be introduced by a internal wave field.

Acknowledgments

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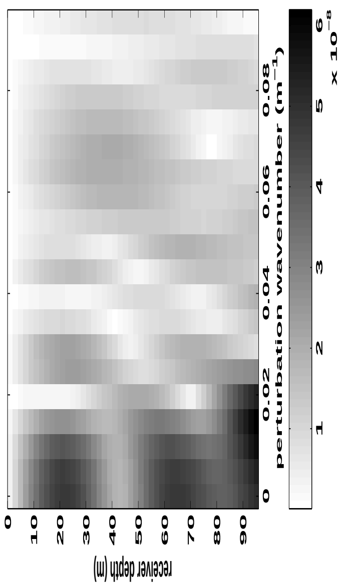

Fig. 2. Effect of bottom-refractive index plane wave perturbations on 20 Hz environmental pressure derivatives, as a function of perturbation wavenumber and receiver depth. Pressure units are in Pa.

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