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Image thresholding using two-dimensional Tsallis–Havrda–Charvát entropy

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Abstract

In this paper, we present a thresholding technique based on two-dimensional Tsallis–Havrda–Charvát entropy. The effectiveness of the proposed method is demonstrated by using examples from the real-world and synthetic images.
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Keywords: Image segmentation; Thresholding; Tsallis–Havrda–Charvát entropy

1. Introduction

Image segmentation is one of the most difficult, dynamic and challenging problems in the image processing domain. Image segmentation denotes a process by which a raw input image is partitioned into non-overlapping regions such that each region is homogeneous and the union of two adjacent regions is heterogeneous. In the literature, many techniques have been developed for image segmentation. Thresholding (see Abutaleb, 1989; Brink, 1992; Chang et al., 1995; Esquef et al., 2002; Kapur et al., 1985; Pavešić and Ribarić, 2000; Portes de Albuquerque et al., 2004; Sahoo et al., 1988, 1997a,b; Sahoo and Arora, 2004; Wong and Sahoo, 1989, and references therein) is a well-known technique for image segmentation. Thresholding is the operation of converting a multi-level image into a binary image. It involves the basic assumption that the objects and the background in the digital image have distinct gray level distributions. If this assumption holds, the gray level histogram contains two or more distinct peaks and threshold values separating them can be obtained. Segmentation is performed by assigning regions having gray levels below the threshold to the background, and assigning those regions having gray levels above the threshold to the objects, or vice versa. Threshold selection methods can be classified into two groups, namely, global methods and local methods (see Sahoo et al., 1988). A global thresholding technique thresholds the entire image with a single threshold value obtained by using the gray level histogram of the image. Local thresholding methods partition the given image into a number of subimages and determine a threshold for each of the subimages. Global thresholding methods are easy to implement and are computationally less involved. They have found many applications in the field of image processing.

In this paper, we propose an automatic global thresholding technique based on two-dimensional Tsallis–Havrda–Charvát entropy. This work is based on the entropic thresholding method proposed in our earlier paper (see Sahoo and Arora, 2004). This new method extends a method due to Pavešić and Ribarić (2000) and Portes de Albuquerque et al. (2004) (see also Esquef et al., 2002). This method includes a previously proposed well-known global thresholding method due to Abutaleb (1989) and

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Brink (1992). This paper is organized as follows: In Section 2, we present a brief description of the Tsallis–Havrda–Charvát entropy. In Section 3, we describe the mathematical setting of the threshold selection and the newly proposed thresholding method. In Section 4, we report the effectiveness of our thresholding method when applied to some real-world and synthetic images. In Section 5, we present some concluding remarks about our method.

2. Tsallis–Havrda–Charvát entropy

Let

$$P = (p_1, p_2, \ldots, p_n) \in \Delta_n,$$

where

$$\Delta_n = \left\{ (p_1, p_2, \ldots, p_n) \,|\, p_i \geqslant 0,\ i = 1, 2, \ldots, n, n \geqslant 2, \sum_{i=1}^{n} p_i = 1 \right\}$$

is a set of discrete finite n-ary probability distributions. Havrda and Charvát (1967) defined an entropy of degree α as

$$H_n^{\alpha}(P) = \frac{1}{1 - 2^{1-\alpha}} \left[1 - \sum_{i=1}^{n} p_i^{\alpha} \right], \tag{1}$$

where α is a real positive parameter not equal to one. Independently Tsallis (1988), proposed a one parameter generalization of the Shannon entropy as

$$H_n^{\alpha}(P) = \frac{1}{\alpha - 1} \left[1 - \sum_{i=1}^{n} p_i^{\alpha} \right], \tag{2}$$

where α is a real positive parameter and $\alpha \neq 1$. Both these entropies essentially have the same expression except the normalizing factor. The Havrda and Charvát entropy is normalized to 1. That is, if $P = (0.5, 0.5)$, then $H_2^{\alpha}(P) = 1$ where as the Tsallis entropy is not normalized. For our application both the entropies yield the same result and we called these entropies as the Tsallis–Havrda–Charvát entropy. However, we use (2) as the Tsallis–Havrda–Charvát entropy. Since $\lim_{\alpha \to 1} H_n^{\alpha}(P) = H_n^1(P)$, the Tsallis–Havrda–Charvát entropy $H_n^{\alpha}(P)$ is a one parameter generalization of the Shannon entropy $H_n^1(P)$, where

$$H_n^1(P) = -\sum_{i=1}^{n} p_i \ln p_i. \tag{3}$$

When $\alpha = 2$, the Tsallis–Havrda–Charvát entropy becomes the Gini–Simpson index of diversity,

$$H_n^2(P) = \left[1 - \sum_{i=1}^{n} p_i^2 \right]. \tag{4}$$

The Tsallis–Havrda–Charvát entropy was studied as the entropy of degree α for the first time by Havrda and Charvát (1967) and then by Daróczy (1970). This entropy became often used in statistical physics after the seminal work of Tsallis (1988).

3. The proposed method

Let $f(m,n)$ be the gray value of the pixel located at the point (m,n). A digital image of size $M \times N$ is a matrix of the form

$$[f(m,n) | m = 1, 2, \ldots, M \text{ and } n = 1, 2, \ldots, N]$$

or in short form $[f(m,n)]$.

Let G denote the set of all gray values $\{0, 1, 2, \ldots, 2^{\ell} - 1\}$. Usually ℓ is either 8 or 16. In our test images, ℓ is 8. Hence for our test images, the set $G = \{0, 1, 2, \ldots, 255\}$. In the next three subsections, we discuss (1) how to construct the two-dimensional histogram, (2) how to compute the probability distributions of object and background classes, and (3) the choice of the entropic criterion function.

3.1. Two-dimensional histogram

In order to compute the two-dimensional histogram of a given image we proceed as follows. Calculate the average gray value of the neighborhood of each pixel. Let $g(x,y)$ be the average of the neighborhood of the pixel located at the point (x,y). The average gray value for the 3×3 neighborhood of each pixel is calculated as

$$g(x,y) = \left\lfloor \frac{1}{9} \sum_{i=-1}^{1} \sum_{j=-1}^{1} f(x+i, y+j) \right\rfloor, \tag{5}$$

where $\lfloor r \rfloor$ denotes the integer part of the number r. While computing the average gray value, disregard the two rows from the top and bottom and two columns from the sides. The pixel's gray value, $f(x,y)$, and the average of its neighborhood, $g(x,y)$, are used to construct a two-dimensional histogram using

$$h(m,n) = \text{Prob}(f(x,y) = m \text{ and } g(x,y) = n), \tag{6}$$

where $m, n \in G$. For a given image, there are several methods to estimate this density function. One of the most frequently used methods is the *method of relative frequency*. The normalized histogram is approximated by using the formula

$$\hat{h}(m,n) = \frac{\text{number of elements in the event } (f(x,y) = m) \text{ and } (g(x,y) = n)}{\text{number of elements in the sample space}} = \frac{\text{number of pixels with gray value } m \text{ and average gray value } n}{\text{number of pixels in the image}}.$$

The joint probability mass function, $p(m,n)$ is given by

$$p(m,n) = \hat{h}(m,n), \tag{7}$$

where $m, n = 0, 1, \ldots, 255$.

3.2. Computation of the object and background probabilities

The threshold is obtained through a vector (t,s) where t, for $f(x,y)$, represents the threshold of the gray level of the pixel and s, for $g(x,y)$, represents the threshold of the

average gray level of the pixel's neighborhood. Using the joint probability mass function $p(m,n)$, a surface can be drawn that will have two peaks and one valley. The object and background correspond to the peaks and can be separated by selecting the vector (t,s) that maximizes a suitable criterion function $\Phi_\alpha(t,s)$. Using this vector (t,s), the domain of the histogram is divided into four quadrants (see Fig. 1).

We denote the first quadrant by $[t+1,255]\times[0,s]$, the second quadrant by $[0,t]\times[0,s]$, the third quadrant by $[0,t]\times[s+1,255]$, and the fourth quadrant by $[t+1,255]\times[s+1,255]$. Since two of the quadrants, first and third, contain information about edges and noise alone, they are ignored in the calculation. Because the quadrants which contain the object and the background, second and fourth, are considered to be independent distributions, the probability values in each case must be normalized in order for each of the quadrants to have a total probability equal to 1. Our normalization is accomplished by using a posteriori class probabilities, $P_2(t,s)$ and $P_4(t,s)$, where

$$P_2(t,s)=\sum_{i=0}^{t}\sum_{j=0}^{s}p(i,j)\quad\text{and}\quad P_4(t,s)=\sum_{i=t+1}^{255}\sum_{j=s+1}^{255}p(i,j). \tag{8}$$

We assume that the contribution of the quadrants which contains the edges and noise is negligible, hence we approximate $P_4(t,s)$ as $P_4(t,s)\approx 1-P_2(t,s)$.

Thresholding an image at a threshold t is equivalent to partitioning the set G into two disjoint subsets:

$$G_0=\{0,1,2,\ldots,t\}\quad\text{and}\quad G_1=\{t+1,t+2,\ldots,255\}.$$

The two probability distribution associated with these sets are:

$$\left(\frac{p(0,0)}{P_2(t,s)},\ldots,\frac{p(0,s)}{P_2(t,s)},\frac{p(1,0)}{P_2(t,s)},\ldots,\frac{p(t,s)}{P_2(t,s)}\right)$$

and

$$\left(\frac{p(t+1,s+1)}{P_4(t,s)},\ldots,\frac{p(t+1,255)}{P_4(t,s)},\frac{p(t+2,s+1)}{P_4(t,s)},\ldots,\frac{p(255,255)}{P_4(t,s)}\right),$$

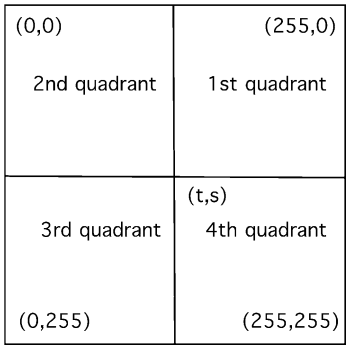

Fig. 1. Quadrants in the 2D histogram due to thresholding at (t,s).

respectively. Based on our previous discussion we have approximated $P_4(t,s)$ as $1-P_2(t,s)$, that is $P_4(t,s)\approx 1-P_2(t,s)$.

3.3. Entropic criterion function

Thus the Tsallis–Havrda–Charvát entropies associated with object and background distributions are given by

$$H_b^\alpha(t,s)=\frac{1}{\alpha-1}\left[1-\sum_{i=0}^{t}\sum_{j=0}^{s}\left(\frac{p_{(i,j)}}{P_2(t,s)}\right)^\alpha\right] \tag{9}$$

and

$$H_w^\alpha(t,s)=\frac{1}{\alpha-1}\left[1-\sum_{i=t+1}^{255}\sum_{j=s+1}^{255}\left(\frac{p_{(i,j)}}{1-P_2(t,s)}\right)^\alpha\right]. \tag{10}$$

The a priori Tsallis–Havrda–Charvát entropy of an image is given by

$$H_T^\alpha=\frac{1}{\alpha-1}\left[1-\sum_{i=0}^{255}\sum_{j=0}^{255}p(i,j)^\alpha\right], \tag{11}$$

where $\alpha\ (\neq 1)$ is a positive real parameter.

If we consider that a physical system can be decomposed into two statistical independent subsystems A and B, the probability of the composite system is $p^{A+B}=p^Ap^B$, then the Tsallis–Havrda–Charvát entropy of the system follows the non-additivity rule

$$H_T^\alpha(A+B)=H_T^\alpha(A)+H_T^\alpha(B)+(1-\alpha)H_T^\alpha(A)H_T^\alpha(B). \tag{12}$$

Thus, in this paper we use

$$\Phi_\alpha(t,s)=H_b^\alpha(t,s)+H_w^\alpha(t,s)+(1-\alpha)H_b^\alpha(t,s)H_w^\alpha(t,s), \tag{13}$$

as a criterion function. We obtain our optimal threshold pair $(t^*(\alpha),s^*(\alpha))$ by maximizing the above criterion function $\Phi_\alpha(t,s)$. Thus

$$(t^*(\alpha),s^*(\alpha))=\operatorname{Arg}\max_{(t,s)\in G\times G}\Phi_\alpha(t,s). \tag{14}$$

4. Analysis of test results

In this section, we discuss the experimental results obtained using the proposed method. This discussion includes the choice of the optimal threshold and the presentation of the optimal threshold values of some real-world and synthetic images. These images are *cameraman.tif*, *kids.tif*, *rice.tif*, *tire.tif*, *bonemarr.tif*, *boats.tif*, *bridge.tif*, *columbia.tif*, *face.tif*, and *lena.tif* and they are shown in Figs. 2–11. The optimal threshold value was computed by the proposed method for these 10 images. Table 1 lists the optimal threshold values that are found for these images for α values equal to 0.3, 0.5, 0.7, 0.8, and 1.0, respectively.

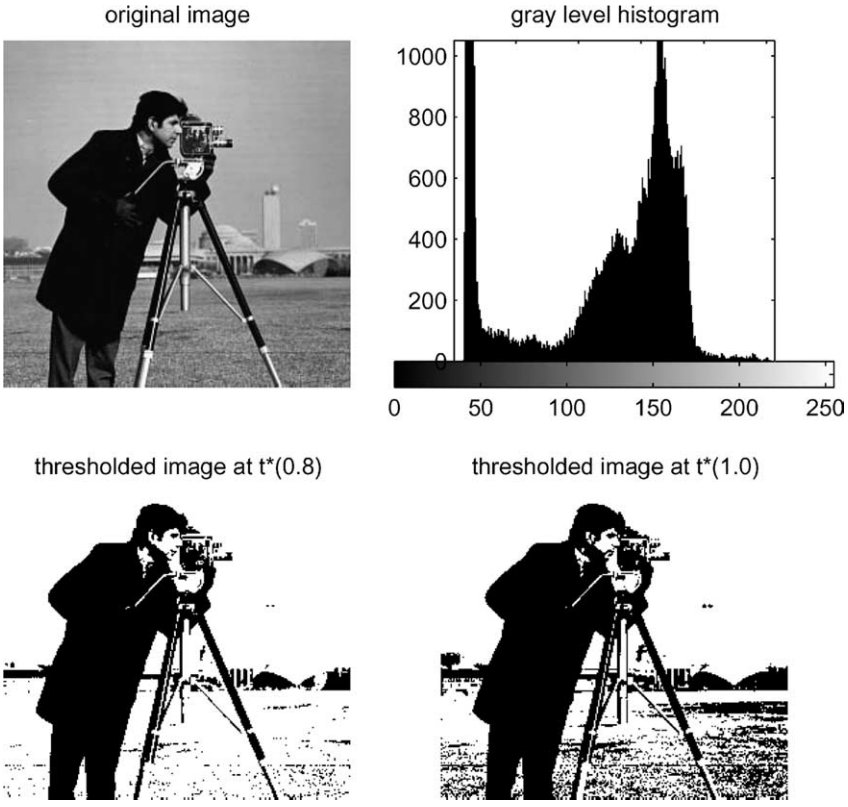

Fig. 2. Cameraman image, its gray level histogram and the thresholded images.

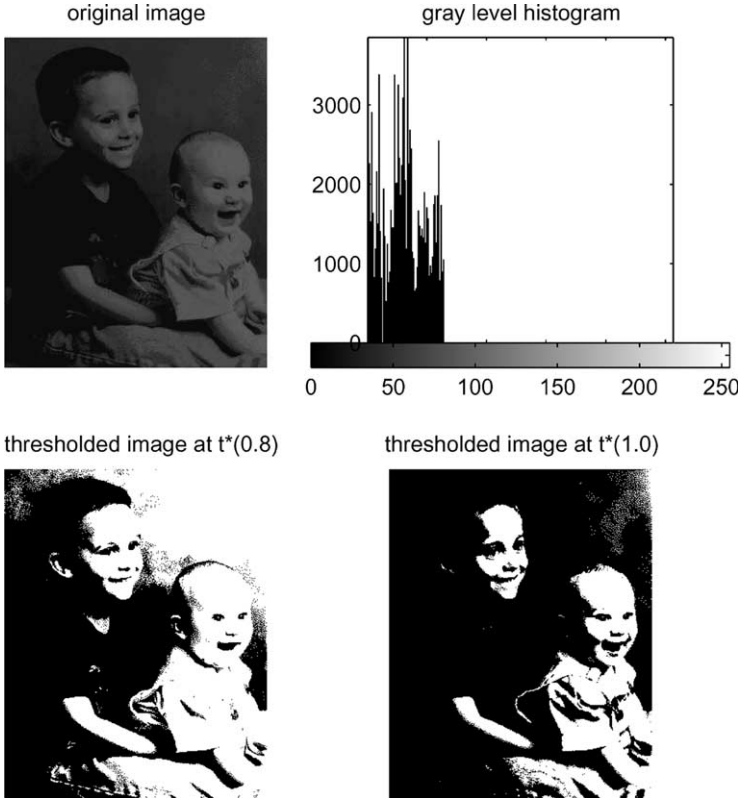

Fig. 3. Kids image, its gray level histogram and the thresholded images.

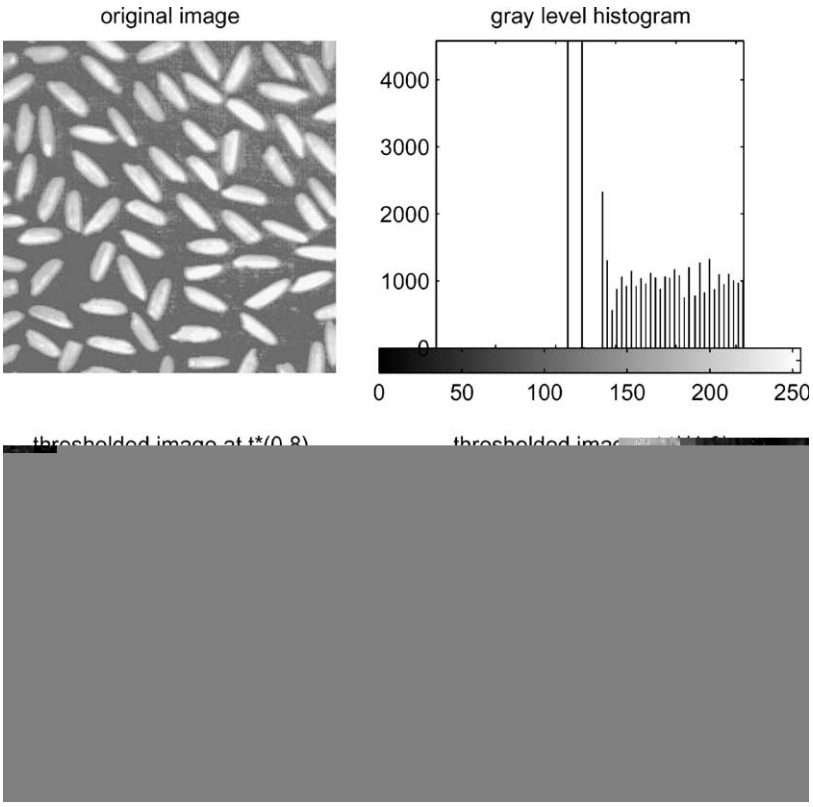

Fig. 4. Rice image, its gray level histogram and the thresholded images.

Fig. 5. Tire image, its gray level histogram and the thresholded images.

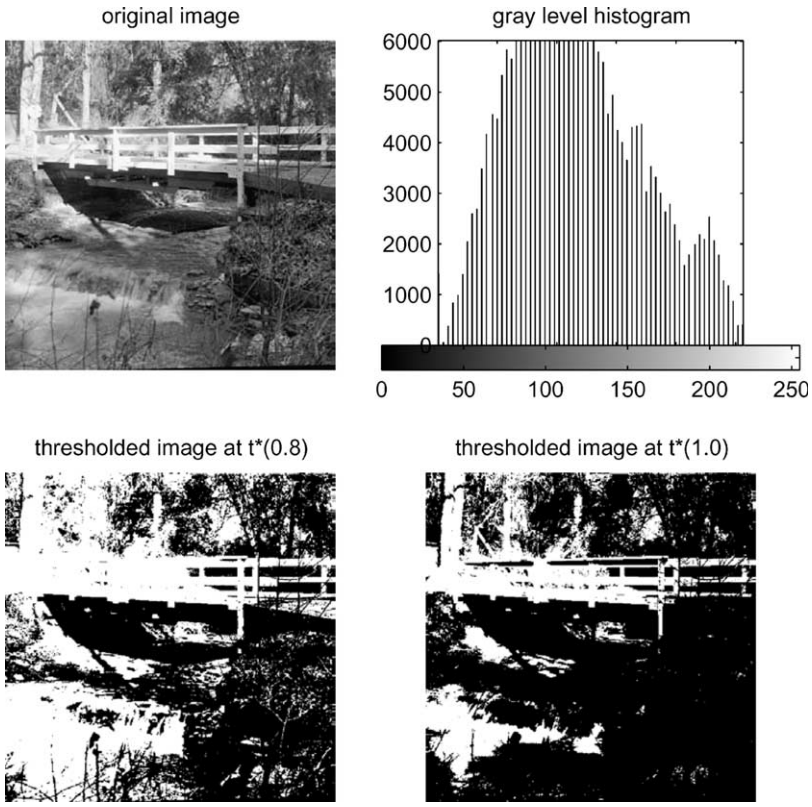

Fig. 8. Bridge image, its gray level histogram and the thresholded images.

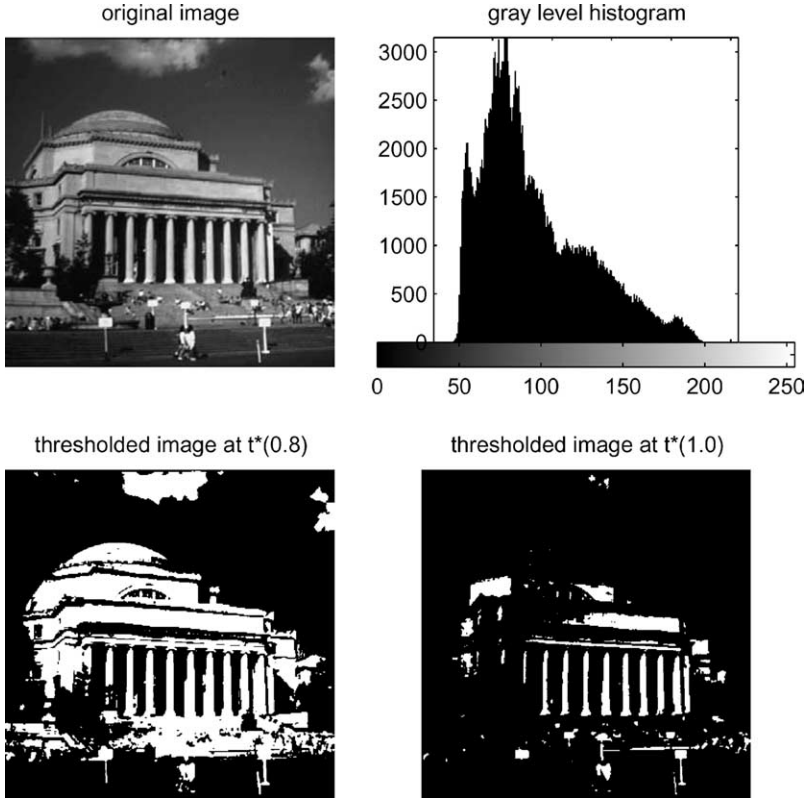

Fig. 9. Columbia image, its gray level histogram and the thresholded images.

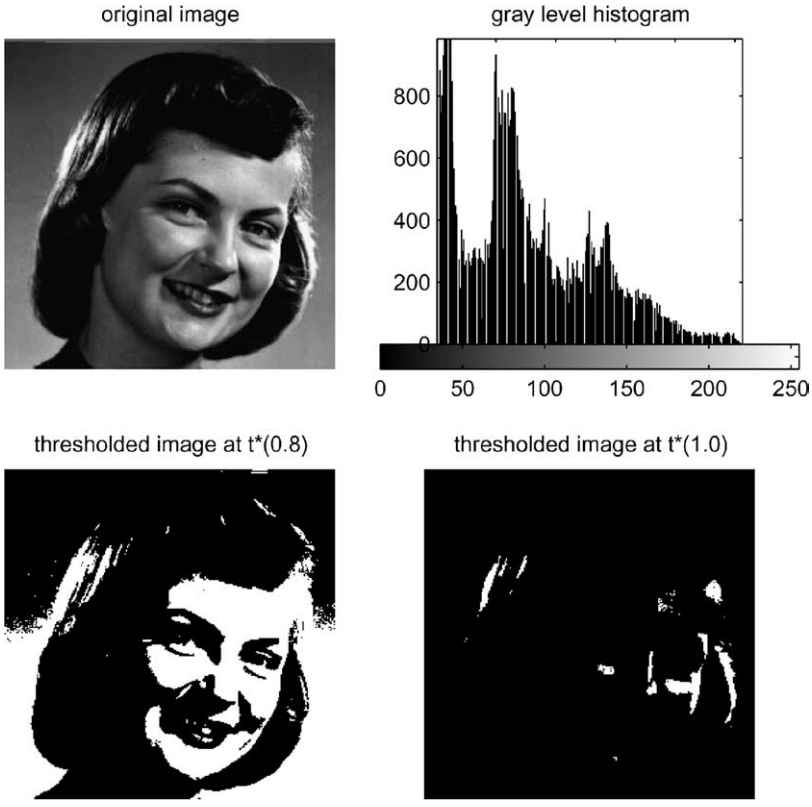

Fig. 10. Face image, its gray level histogram and the thresholded images.

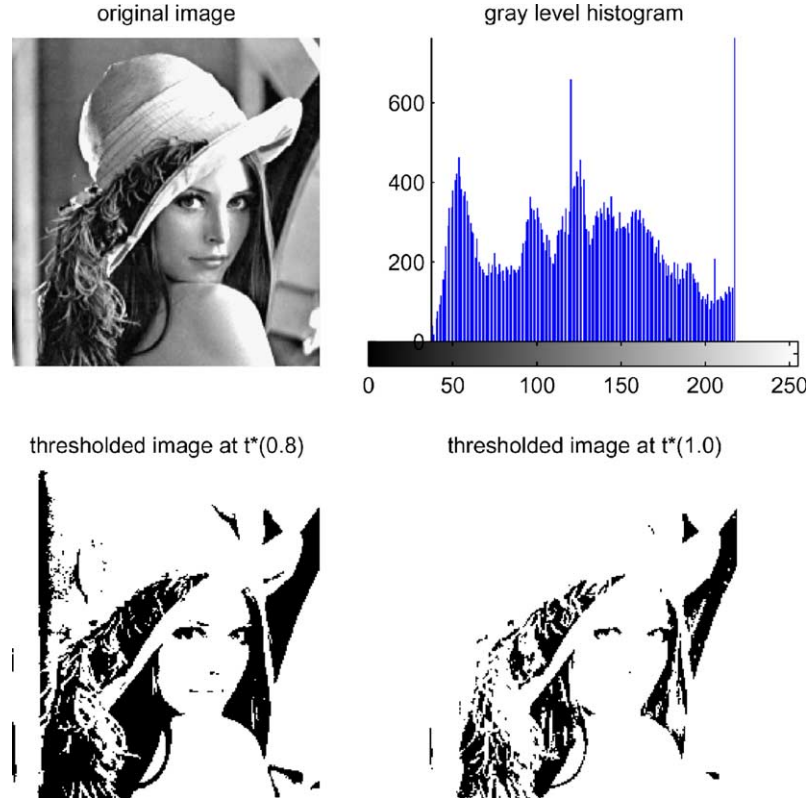

Fig. 11. Lena image, its gray level histogram and the thresholded images.

Table 1
The optimal threshold values for various values of α

Test image	$t^\star(0.3)$	$t^\star(0.5)$	$t^\star(0.7)$	$t^\star(0.8)$	$t^\star(1.0)$
camaraman.tif	99	99	99	95	119
kids.tif	32	32	32	29	11
rice.tif	182	182	182	182	178
tire.tif	111	111	93	93	55
bonemarr.tif	122	121	121	121	216
boats.tif	102	103	103	103	158
bridge.tif	125	130	125	113	154
columbia.tif	122	120	120	103	160
face.tif	104	104	104	95	204
lena.tif	112	112	106	80	38

Using the original test image $[f(m,n)]$, the thresholded image $[f_t(m,n)]$ was computed using formula

$$f_t(m,n) = \begin{cases} 0 & \text{if } f(m,n) \leqslant t^\star(\alpha), \\ 1 & \text{if } f(m,n) > t^\star(\alpha). \end{cases}$$

The original images together with their histograms and the thresholded images obtained by using the optimal threshold value $t^\star(0.8)$ and $t^\star(1.0)$ are displayed side by side in Figs. 2–11. Note that the thresholded image obtained by threshold value $t^\star(1.0)$ is same as the one obtained by Abutaleb's method (see Abutaleb, 1989).

4.1. The best value for the parameter α

Our analysis is based on how much information is lost due to thresholding. In this analysis, given two thresholded images of a same original image, we prefer the one which lost the least amount of information. Using the above ten images and also many other images, we conclude that when α is 0.8 our proposed method produced the best optimal threshold values. When α was greater than 1, this proposed method did not produce good threshold values. In fact the threshold values produced were unacceptable. When the value of α was one, the threshold value produced was not always a good threshold value (see Figs. 2–11). This new method performs better than Abutaleb's method when $\alpha = 0.8$.

4.2. The window size for computing $g(x,y)$

In this method of thresholding, we have used in addition to the original gray level function $f(x,y)$, a function $g(x,y)$ that is the average gray level value in a 3×3 neighborhood around the pixel (x,y). This approach can be extended to an image pyramid, where an image on the next higher level is composed of average gray level values computed for disjoint 3×3 squares. In order to find out which pyramid level will produce an optimal threshold value, we have tested all the test images with different neighborhood sizes and different α values. In particular, we have considered neighborhood sizes of 2×2, 3×3, 4×4, 5×5, 8×8 and 16×16, respectively. Neighborhood sizes of 3×3, 4×4, 5×5 and 8×8 all produced similar optimal threshold values giving acceptable binary images. From the point of view of computational time and image quality, a neighborhood size of 3×3 or 5×5 with α value equals 0.8 would be ideal for thresholding with this proposed method.

5. Conclusion

Using Tsallis–Havrda–Charvát's entropy of degree α, we have developed a new method for thresholding of images. This method usages a two-dimensional histogram computed from the image. The two-dimensional histogram was constructed using the gray value and local average gray value to choose an optimal threshold value.

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