# TRAVELING SALESMAN PROBLEMS WITH PROFITS: AN OVERVIEW

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**Abstract**

Traveling Salesman Problems with Profits (TSPs with Profits) are a generalization of the Traveling Salesman Problem (TSP) where it is not necessary to visit all vertices. With each vertex is associated a profit. The objective is to find a route with a satisfying collected profit (maximized) and travel cost (minimized). Applications of these problems arise in contexts such as traveling salesman problems, job scheduling or carrier transportation. In this paper, the existing literature about TSPs with Profits is surveyed.

**Keywords**

Traveling salesman problem, quota tsp, profitable tour problem, prize-collecting tsp, survey.

## I. Introduction

The Traveling Salesman problem (TSP) and the Vehicle Routing Problem (VRP) are among the most widely studied combinatorial optimization problems. They deal with optimally visiting customers from a central depot. Many extensions of these problems are encountered in the literature. The reader is referred to the survey papers of Laporte (1992a) and (1992b) for a detailed literature review. A special case of the TSP occurs when customers are selected according to a profit collected by visiting them. This feature gives rise to a number of problems that we gather together under the name of Traveling Salesman Problems with Profits (TSPs with Profits).

Let $G = (V, A)$ be a complete graph where $V = \{v_1, \ldots, v_n\}$ is a set of $n$ vertices and $A$ is the set of arcs (directed TSPs with Profits) or edges (undirected TSPs with Profits). Let us associate a profit $p_i \geq 0$ with each vertex $v_i \in V$ (with $p_1 = 0$) and a distance $c_{ij} \geq 0$ with each arc or edge $(v_i, v_j) \in A$. In the following, we interpret vertices as customers and distances as travel costs. Vertex $v_1$ is interpreted as a depot. The travel cost matrix is assumed to satisfy the triangle inequality. When the travel cost matrix is symmetrical we fall into the undirected case. TSPs with Profits consist in determining an elementary circuit including vertex $v_1$, but not necessarily all the other vertices, with an interest in both the collected profit and the travel costs.

This class of problems might be seen as a bicreteria traveling salesman problem with two opposite objectives, one pushing the salesman into traveling (to collect profits) and the other inciting him to minimize travels (with the right to drop vertices). Readers interested in other kinds of multicriteria traveling salesman problems, and more generally combinatorial optimization problems, are referred to Ehrgott (2000). Viewed in this light, solving TSPs with profits should result in finding a noninferior solution set, i.e., a set of feasible solutions such that no objective can be improved without deteriorating the other. Actually, most researchers who face these problems are interested in monocriterion versions. Thus, either the two objectives are weighted and combined linearly, or one of the objectives is constrained to a specified value. This defines three classes of problems, which we gather together under the name of TSPs with Profits. To our knowledge, the only attempts to solve the complete biobjective problem are due to Keller (1985) and Keller and Goodchild (1988), who name it the Multiobjective Vending Problem, but the solution approaches finally consist in sequentially solving monocriterion versions of the problem.

Let us now detail the three classes of problems that make up TSPs with Profits:

- When the aim is to find a circuit that maximizes collected profit and such that travel costs do not exceed a preset value $c_{max}$, we are in the class of the Selective Traveling Salesman Problem (STSP).
- When the aim is to find a circuit that minimizes travel costs and whose collected profit is not smaller than a preset value $p_{min}$, we are in the class of the Quota Traveling Salesman Problem (Quota TSP).
- When both objectives are combined in the objective function, we are in the class of the Profitable Tour Problem (PTP). In the following, we consider that the PTP consists in minimizing travel costs minus profit. This objective function could be more naturally written as a maximized profit minus travel costs but we prefer the previous writing as it is consistent with the TSP, especially for the meaning of upper or lower bounds. The PTP is often encountered in extensions including another constrained resource. The generic name is then Traveling Salesman Subset-Tour Problem with One Additional Constraint (TSSP+1). When this resource expresses a need, we are in an important special case known under the name of Prize-collecting Traveling Salesman Problem (PCTSP). In this paper, we review the existing literature about these problems, except the STSP that has been the subject of nu-

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merous articles and to which a whole survey is devoted (see Feillet *et al.* 2000a). Even so, for greater convenience, a short section mentions the main situations where the STSP arises and the main solution approaches dedicated to its solution.

As indicated previously, we classify each problem in one of these three classes under the assumptions that both the profit vector and the travel cost matrix are positive and that the travel cost matrix satisfies the triangle inequality. As some papers deal with data out of this framework (e.g., negative travel costs), we have to precisely explain how we settle formulations in these cases.

Let us first suppose that the objective is to minimize travel costs minus profit (PTP situation) and that the profit vector or the travel cost matrix do not satisfy the expected conditions. The addition of any constant simultaneously to every cost $c_{ij}$, with $(v_i, v_j) \in A$, and every profit $p_i$, with $v_i \in V$, does not change the feasible solution values, as solutions are elementary circuits. Actually, we prefer to keep $p_1 = 0$, but this does not alter the relative ranking among the feasible solutions. Then, one can easily choose a value that makes the travel cost matrix positive and satisfying the triangle inequality. If, in this situation, some vertices have a negative profit, they can be excluded from the vertex set, since they are not worth routing. Similar considerations are found in Fischetti and Toth (1988).

Let us now suppose that we face a problem where the objective is to minimize travel costs (Quota TSP situation), and that the travel cost matrix does not meet the expected requirements. We then proceed in the following way. We start by changing the profit naming, in order to be in the situation of a problem with a new constrained resource and profits set to zero. Thus, we just have to apply the previous transformation scheme to enforce the requirements and to classify this problem as a PTP with an additional resource.

We now have to insist on the similarity of the various TSPs with Profits. A first point concerns solution procedures. Even though most of the solution procedures focus on a specific problem, many might be efficient for related problems. A typical example is given by procedures solving the PCTSP, which generally stay as fast and effective when applied to instances with profit vector set to zero, that is instances of the Quota TSP (see section V-C). A second point concerns solution values. TSPs with Profits are closely related, being different monocriterion versions of a single biobjective problem (the multiobjective vending problem). The noninferior solution set of the biobjective problem is easy to compute in a sequential way. Stating an objective as a constraint and solving many instances with sequential limit values results in finding this solution set. In the same way, this set can be at least partially computed when the objective function combines both objectives by solving many instances changing the trade-off relationship between the objectives. Thus, it is always possible to tackle a problem by sequentially solving a different one, for which an efficient and effective procedure would be available. These considerations are mentioned in Keller and Goodchild (1988), Keller (1989) or Pekny *et al.* (1990). Further results might be found in Keller (1985).

All the problems presented in this paper trivially belong to the class of NP-hard problems (see Fichetti and Toth 1988, Pekny *et al.* 1990 and Mittenthal and Noon 1992 for example). In particular, most of them contain as a particular case the TSP.

In the following, we classify TSPs with Profits according to their objective function. Section II deals with linear programming formulations and structural properties. In section III, we present the Quota TSP. Section IV describes the PTP, while section V describes profitable tour problems with an additional resource. Other extensions of the PTP are found in section VI. The last section VII concerns the STSP and just mentions some main research works.

## II. Integer linear programming formulations

As TSPs with Profits are closely related to the TSP, most of the works devoted to structural properties of TSPs with Profits are inspired by similar results for the TSP. Some results concerning facet defining inequalities will be described below, but we first focus on encountered integer linear programming formulations. Both in directed and undirected cases, they mostly follow from integer linear programming formulations for the TSP. As the variety of problems gathered together in the class of TSPs with Profits does not induce such a variety of formulations, we describe them simultaneously.

### A. Formulations

Let us begin with the directed case. We propose a possible formulation framework. Most of the formulations found in the literature are structured in accordance with this framework (e.g., Gensch 1978, Fischetti and Toth 1988 or Balas 1989). We associate one binary variable $x_{ij}$ to every arc $(v_i, v_j) \in A$, equal to 1 if and only if the corresponding arc is used in the solution, and one binary variable $y_i$ to every vertex $v_i \in V$, equal to 1 if and only if the corresponding vertex is visited.

In the case of the PTP, the formulation is

$$\text{minimize} \sum_{v_i \in V} \sum_{v_j \in V \setminus \{v_i\}} c_{ij} x_{ij} - \sum_{v_i \in V} p_i y_i \tag{1}$$

subject to

$$\sum_{v_j \in V \setminus \{v_i\}} x_{ij} = y_i \qquad (v_i \in V), \tag{2}$$

$$\sum_{v_i \in V \setminus \{v_j\}} x_{ij} = y_j \qquad (v_j \in V), \tag{3}$$

$$\textit{subtour elimination constraints,} \tag{4}$$

$$y_1 = 1, \tag{5}$$

$$x_{ij} \in \{0,1\} \qquad ((v_i, v_j) \in A) \tag{6}$$

$$y_i \in \{0,1\} \qquad (v_i \in V) \tag{7}$$

Constraints (2), (3) are the so-called assignment constraints. Constraints (4) eliminate subtours which do not involve the depot. These constraints are detailed in the next subsection.

For the Quota TSP, the objective function (1) should be replaced with

$$\text{minimize} \sum_{v_i \in V} \sum_{v_j \in V \setminus \{v_i\}} c_{ij} x_{ij} \tag{8}$$

while constraint (9) is added

$$\sum_{v_i \in V} p_i y_i \geq p_{min} \tag{9}$$

For a structural study purpose, it is interesting to note that constraint (9) can be replaced by a knapsack type constraint involving non visited vertices.

For both problems, some slight changes on variables definition lead to similar formulations. It is possible to let $y_i$ variables remain implicit (e.g., Gensch 1978). Packing constraints then replace assignment constraints (2) and constraints (10) are added

$$\sum_{v_i \in V} x_{ik} = \sum_{v_j \in V} x_{kj} \qquad (v_k \in V) \tag{10}$$

while one might suppress constraints (3), which turn out to be redundant. It is also possible to consider non-visited vertices as separate subtours of length 1 (self-loops) by introducing variables $x_{ii}$ instead of variables $y_i$. This approach is proposed by many researchers (e.g., Gensch 1978 or Fischetti and Toth 1988). It is convenient to emphasize the assignment problem substructure. It is also important to report that this formulation is used to define and study polytopes related to TSPs with Profits (see section II-C). In a different manner, Fischetti and Toth (1988) propose to work in an augmented graph. A dummy vertex $v_0$ is introduced, to which non-visited vertices are connected (arc variables $x_{0i}$ replace variables $y_i$). This approach points out to the underlying arborescence problem substructure of the TSPs with Profits.

Let us now consider the undirected case. For every vertex subset $S$, let $\delta(S)$ be the set of edges with one end in $S$ and the other end in $V \setminus S$. In order to respect usual notations, we rename for a short while $E$ the set of edges. Bienstock *et al.* (1993) propose the following formulation for the PTP:

$$\text{minimize} \sum_{e \in E} c_e x_e - \sum_{v_i \in V} p_i y_i \tag{11}$$

subject to

$$\sum_{e \in \delta(\{v_i\})} x_e = 2y_i \qquad (v_i \in V), \tag{12}$$

$$\textit{subtour elimination constraints,} \tag{13}$$

$$y_1 = 1, \tag{14}$$

$$x_e \in \{0,1\} \qquad (e \in E), \tag{15}$$

$$y_i \in \{0,1\} \qquad (v_i \in V) \tag{16}$$

In this formulation, cycles of length 2 are not allowed. Fischetti *et al.* (1998) claim that one can assume, without loss of generality, that the optimal cycle contains at least 3 edges, as cycles of length 2 can easily be found by enumeration. Laporte and Martello (1990) propose to widen domain of variables $x_e$ to $\{0,1,2\}$ for every $e \in E$ connected to the depot, so as to consider cycles of length 2 in the model.

At last, let us point out that a completely different formulation, involving permutation and flow variables, is proposed by Millar and Kiragu (1997) for the STSP.

### B. Subtour elimination constraints

Several formulations are proposed for subtour elimination constraints (4) in the directed case. They mostly are straightforward adaptations of existing subtour elimination constraints for the TSP, taking into account the presence of a subtour involving vertex $v_1$. Fischetti and Toth (1988) propose

$$\sum_{v_i \in S} \sum_{v_j \in V \setminus S} x_{ij} \geq y_k \qquad (S \subset V, v_1 \in S, v_k \in V \setminus S) \tag{17}$$

or

$$\sum_{v_i \in S} \sum_{v_j \in S \setminus \{v_i\}} x_{ij} \leq |S| - 1 \qquad (S \subset V \setminus \{v_1\}) \tag{18}$$

Laporte and Martello (1990) propose another kind of formulation, which we rewrite by introducing $y_i$ variables

$$2 \sum_{v_k \in S} y_k \leq |S| \sum_{v_i \in S, v_j \in V \setminus S} (x_{ij} + x_{ij}) \qquad (S \subset V \setminus \{v_1\}, |S| \geq 2) \tag{19}$$

Balas (1989) studies more deeply these constraints. He proposes several valid inequalities related to the subtour elimination constraints for the Traveling Salesman Polytope and conditions for which they are facet defining. This study is initially dedicated to a version of the PCTSP with no depot, but results are detailed for four polytopes, the Profitable Tour Polytope, the Quota Traveling Salesman Polytope (which is also the Prize-collecting Traveling Salesman Polytope), and the two related polytopes where the depot condition is suppressed.

Similar formulations as above are found in the undirected case. Bienstock *et al.* (1993) propose

$$\sum_{e \in \delta(S)} x_e \geq 2y_i \qquad (S \subset V, v_1 \in S, v_i \in V \setminus S) \tag{20}$$

while Leifer and Rosenwein (1994) propose

$$\sum_{e \subset S} x_e \leq |S| - 1 \qquad (S \subset V \setminus \{v_1\}, 3 \leq |S| \leq n-2) \tag{21}$$

Kubo and Kasugai (1992) adapt for the undirected case some of the results proven by Balas (1989) in directed graphs.

### C. Other structural properties and valid inequalities

TSPs with Profits deal with searching a simple cycle in a graph, under certain constraints. Therefore, their polytopes are included in the convex hull of the simple cycles of a graph. This polytope is similar to the Profitable Tour Polytope, apart from the requirement that the depot be included in the cycle. The studies devoted to this polytope directly concern TSP with Profits.

It is named Cycle Polytope in an undirected graph by Coullard and Pulleyblank (1989). It has been heavily studied by Kubo and Kasugai (1992) and Bauer (1997). They propose many facet defining valid inequalities, mostly related to families of facet defining inequalities for the Traveling Salesman Polytope, as clique tree, 2-matching, comb, path or crown inequalities.

In a directed graph, this polytope has also seen a lot of attention by Balas (1995) and Helmberg (1999). Balas (1995) gives a general method for deriving a facet defining inequality from any facet defining inequality for the Asymmetric Traveling Salesman Polytope. This method is applied to several families of facet defining inequalities for the Asymmetric Traveling Salesman Polytope, including comb, odd CAT, SD, clique tree and lifted cycle inequalities. Helmberg (1999) faces the problem with the depot requirement (i.e., the PTP), proposes some facet defining valid inequalities but does not manage to completely adapt Balas' results.

Many of these results are extended to the intersection of this polytope with the Knapsack Polytope, respectively in Bauer *et al.*(1998) for the undirected case and Balas (1995) for the directed case. The polytope is then similar to the Quota Traveling Salesman Polytope (and Prize Colecting Traveling Salesman Polytope), apart from the requirement that the depot be included in the cycle. Other facet defining inequalities are derived from facet defining inequalities for the Knapsack Polytope (Bauer *et al.* 1998, Balas 1989). Let us also point out that Balas (1989) clearly settles and strengthens all results about facet defining inequalities derived from the Knapsack Polytope for the situation with the depot requirement, i.e., exactly for the Quota TSP situation. Inclusion relations between the different previously mentioned polytopes are summarized in figure 1. $P_1 \rightarrow P_2$ means polytope $P_2$ is included in polytope $P_1$.

## III. The Quota TSP

The Quota TSP is introduced by Awerbuch *et al.* (1998) on an undirected graph $G$. A preset value $p_{min}$ is given. The objective is to find an elementary cycle including the depot in graph $G$ that minimizes travel costs and whose collected profit is not smaller than $p_{min}$.

Awerbuch *et al.* (1998) do not introduce this problem in connection with a practical application, but they propose the following interpretation. A salesman must sell a quota $p_{min}$ of brushes. He owns a map with city locations, distances between cities and knows how many brushes he could sell in each city. His aim is to travel along a route while minimizing the traveled distance and selling the quota of brushes. For this problem, the authors develop an approximation algorithm with performance guarantee, which we present below. Quite surprisingly, we are not aware of any other paper clearly focused on the Quota TSP. Even so, many solution procedures dedicated to the PCTSP (both in the directed and the undirected cases) can be efficiently applied to it (see section V-C for details).

Awerbuch *et al.* (1998) achieve an approximation algorithm for the Quota TSP through an approximation algorithm for the $k$-Minimum-Spanning-Tree problem ($k$-MST problem). The $k$-MST problem consists in finding a tree of least weight (distance) that spans exactly $k$ vertices on the graph. For the Quota TSP solution, the authors propose to replace each vertex $v_i \in V$ with $p_i$ copies of itself at the same location and to compute an approximate solution of the $k$-MST problem in this graph, with $k = p_{min}$. It just remains to classically double the computed tree to obtain a tour.

Thus, the core subject of their paper is an $O(log^2 k)$ approximation algorithm for the $k$-MST problem. This algorithm is based on the determination of clusters. The pair of clusters $(C_i, C_j)$, initially vertices, that minimizes $\frac{c_{ij}}{min(|C_i|,|C_j|)}$ is merged into a new cluster ($c_{ij}$ means here distance between clusters $C_i$ and $C_j$, i.e., the minimum distance between components of the clusters). Clusters are merged iteratively in the same manner until a cluster of size at least $\frac{k}{4}$ is found. This cluster is then put aside and the procedure is repeated with $k$ updated and while $k > 0$. All the vertices selected are connected with a shortest spanning tree procedure (the end of the procedure is actually slightly modified to achieve a better approximation). Authors note that this approximation also achieves a ratio of $O(log^2 n)$, which induces that their approximation algorithm achieves a factor $O(log^2(min(n, p_{min})))$ for the Quota TSP.

This field of investigation has seen a huge development in the past few years and several other results have recently improved the approximation ratio for the $k$-MST problem (and consequently for the Quota TSP). Blum *et al.* (1999) achieves a constant-factor approximation. Arora and Karakostas (2000) present an approximation algorithm that computes, for any $\epsilon > 0$, a $(2 + \epsilon)$ approximation in $n^{O(1/\epsilon)}$ time. Arora (1998) proposes an approximation algorithm that computes, for any $\epsilon > 0$, a $(1 + \epsilon)$ approximation for the Euclidean $k$-MST problem in polynomial time.

## IV. The PTP

The objective of the PTP is to find an elementary circuit including the depot in graph $G$ that minimizes travel costs minus profit. The PTP naming has been coined by Dell'Amico *et al.* (1995) even if several papers dealt with it before.

The first attempt to solve it dates back to 1987, with Volgenant and Jonker (1987), under the name of Generalized TSP. They explain how solving a PTP amounts to solving an Asymmetric TSP on a transformed graph having $2n+1$ vertices. Their transformation scheme is detailed below. It allows to solve the PTP with every solution procedure

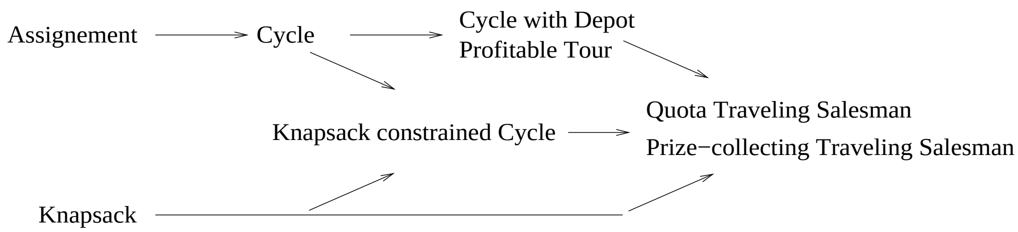


Fig. 1. TSPs with Profits related polytope inclusions.

dedicated to the Asymmetric TSP.

Dell'Amico *et al.* (1995) propose two bounding schemes for the PTP:

- First, they use the previous transformation and derive a bound from the transformed graph by relaxing assignment constraints (2), obtaining somewhat of a 1-arborescence problem. This bound does not manage to reach 65% of the optimal solution value on a set of asymmetric instances.
- Besides, they compute another type of bound by simply relaxing the subtour elimination constraints in the PTP formulation and solving the assignment problem obtained. This second bounding procedure proves to be more effective and gives about 90% of the optimal solution value on the same instances.

Unfortunately, no results are given for symmetric instances.

In the field of approximation algorithms, Bienstock *et al.* (1993) propose a polynomial approximation algorithm having a factor of $\frac{5}{2}$ performance guarantee for the undirected PTP. In a first step of the algorithm, the linear programming relaxation of the problem is solved using the ellipsoid method and vertices $v_i$ with $y_i \geq \frac{3}{5}$ are selected. Then a TSP heuristic with a worst-case performance guarantee is computed on this set of vertices. Theorems show the factor-of-$\frac{5}{2}$ performance of the route computed. Goemans and Williamson (1995) improve the performance guarantee and obtain a $(2 - \frac{1}{n-1})$-approximation algorithm with a purely combinatorial approach.

In the context of a scheduling problem on $m$ non-identical machines with sequence dependent setup times, Helmberg (1999) faces a problem he calls the m-Cost ATSP. It consists in an Asymmetric TSP with $m$ salesmen, where the travel costs of these salesmen are distinct. He reports that the one-machine subproblem is a PTP and studies its polytope (see section II-C). Results are used to analyze the structural properties of the m-Cost ATSP.

### *Transformation of the PTP into an Asymmetric TSP*

Volgenant and Jonker (1987) claim that solving a PTP amounts to solving an Asymmetric TSP in a transformed graph. Let us call $G' = (V', A')$ the transformed graph. The depot is duplicated in a vertex $v_{n+1}$, with a straightforward update of the arc set. Vertices $v_{n+2}, \ldots, v_{2n+1}$ are added, which respectively duplicate vertices $v_2, \ldots, v_{n+1}$. Thus, $V' = V \cup \{v_{n+1}, \ldots, v_{2n+1}\}$. The existing arcs are considered with the same cost coefficients. Other arcs are added:

- $\{(v_{n+i}, v_i), 2 \leq i \leq n\}$, with cost coefficients set to $p_i$,
- $\{(v_i, v_{n+i+1}), 2 \leq i \leq n\}$, with cost coefficients set to 0,
- $\{(v_{n+i}, v_{n+i+1}), 2 \leq i \leq n\}$, with cost coefficients set to 0,
- $(v_{n+1}, v_{n+2})$ and $(v_{2n+1}, v_1)$, with cost coefficients set to 0.

In this graph, non collected profit on a vertex $v_i$ is seen as a penalty paid by visiting the vertex through arcs $(v_{n+i}, v_i)$ and $(v_i, v_{n+i+1})$.

It must be noted that in this transformation, vertices $n+2$ to $2n$ have out-degrees equal to 2 and vertices $n+3$ to $2n+1$ have in-degrees equal to 2. As a consequence, vertices $n+3$ to $2n+1$ must belong to sequences of the form $\{v_{n+k} \rightarrow v_k \rightarrow v_{n+k+1}\}$ (in the case vertex $v_k$ is not visited in the PTP tour) or $\{v_{n+k} \rightarrow v_{n+k+1}\}$ (in the case vertex $v_k$ is visited in the PTP tour). In fact, every solution can be split into two parts; the first one corresponds to the "real" tour being performed and consists in a sequence of original vertices between vertices 1 and $n+1$; the second one accounts for the unvisited vertices that appear between duplicate vertices $n+2$ to $2n+1$.

Finally, solving a PTP results in solving an Asymmetric TSP on a graph augmented with $n+1$ vertices and $3n-1$ arcs. However, no computational results assess the efficiency of this transformation.

## V. The PTP with an additional resource

The PTP is most widely studied in extensions involving a new resource. In the general case, when any single constraint enforces the resource, the problem is known under the name TSSP+1. But many situations involve either a resource expressing a limitation (e.g., for Gensch 1978) or a need. For more convenience, let us name the first case the Knapsack Constrained Profitable Tour Problem (KCPTP). In the second case, we are not aware of any problem dealing with a resource collected on arcs, but when this resource is collected on vertices the problem is well known under the name of Prize-collecting Traveling Salesman Problem (PCTSP). The three next subsections respectively describe papers dealing with the TSSP+1 and papers more precisely focused on either the KCPTP or the PCTSP. The subsection dedicated to the PCTSP is lengthier as this problem has seen a larger amount of interest from the researchers and as it admits both the PTP and the Quota TSP as special cases.

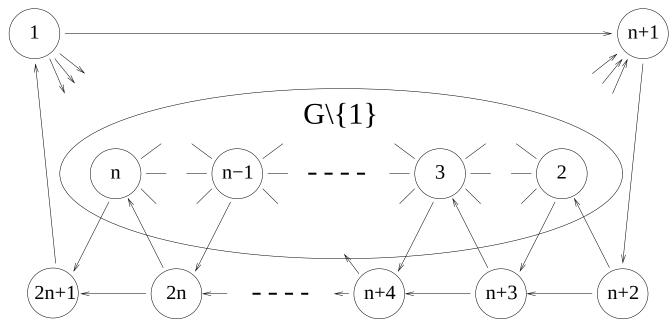


Fig. 2. Transformation of PTP into ASTP.

## *A. The TSSP+1*

In its usual formulation, introduced by Mittenthal and Noon (1992), the TSSP+1 is stated as the problem of finding a minimum cost subtour in a graph under one additional constraint. It involves only this new constraint, which implicitly corresponds to a resource, and profit is not considered. There is also no consideration on the sign of the cost matrix. As explained in section I, we consider this cost matrix as the result of a positive cost matrix satisfying the triangle inequality minus a positive profit vector on vertices. Then, seeing the initial resource as a second resource, we recover the previous description of the PTP with an additional resource, which justifies our classification of the TSSP+1.

An interesting feature of this problem, and of every TSPs with Profits, which distinguishes them from the TSP, is the possibility of improving the solution by dropping a vertex out of it. Mittenthal and Noon (1992) exploit this feature in a simple heuristic algorithm based on insertion and deletion. An interesting application of the TSSP+1 in its general form and of this heuristic procedure is proposed by Noon *et al.* (1994), for which the TSSP+1 serves as a subproblem in a procedure attempting to solve (heuristically) the VRP. Both the TSSP+1 and the VRP dedicated heuristics are described below. More results might be found in Pillai (1992) who dedicated a whole thesis to the TSSP+1.

### A.1 Mittenthal and Noon Insert/Delete procedure

This heuristic is based on insertion and deletion. The procedure begins with an initial not necessarily feasible route (for example a route restricted to the depot, or a route obtained with a TSP solution procedure). Let $T_i$ be a one-city alteration of route $T$. If vertex $v_i$ belongs to route $T$, $T_i$ is the route obtained from $T$ after $v_i$ is deleted. If vertex $v_i$ does not belong to route $T$, $T_i$ is the route obtained from the least cost insertion of $v_i$ in $T$. At each step, if the route is feasible, the candidate vertex $v_i$ so that $T_i$ is feasible with the best improvement on the total travel cost is altered. Otherwise, the vertex $v_i$ so that the infeasibility does not increase with the best improvement on the total cost is altered. A 3-opt tour improvement procedure is proposed as a final step.

Besides its simplicity and its quickness (unfortunately, no computing times are given to be more precise), the strength of this procedure lies in its genericity as many problems are special cases of TSSP+1. The effectiveness of the procedure, however, is far from standing up to specialized heuristics taking into account the particularities of the resource constraint.

### A.2 Noon *et al.* heuristic procedure for the VRP

Noon *et al.* (1994) propose a TSSP+1 decomposition strategy for the VRP. Relaxing the vertex visit constraints in a Lagrangian fashion, they obtain $K$ identical TSSP+1 as subproblems, where $K$ is the number of vehicles. Solving a subproblem gives a lower bound, but does not provide a solution since a single route emerges. The authors avoid this difficulty by initially assigning a vertex to each vehicle. These $K$ vertices are chosen on the border of the convex hull of the vertex set and with a maximal distance, so that the probability for both of them to be in a same VRP optimal route is minimized. The Lagrangian multiplier problem is solved with a subgradient algorithm. At each step, the Insert/Delete heuristic algorithm of section V-A.1 is called to solve the subproblems. Let us point out that this algorithm is well fitted because it is able to manage efficiently negative costs, it accepts enforced vertices and it is able to begin with a given solution (the previous solution obtained during the subgradient algorithm).

This approach proves to be comparable to other recent heuristic algorithms for a set of 12 instances taken from the literature, involving up to 150 customers, but at the price of a much longer computing time (up to 45 minutes on a VAX9000). A new version based on a parallel implementation which permits to reduce computing times is described in Bowers *et al.* (1996).

## *B. The KCPTP*

The KCPTP consists in a TSSP+1 where the additional constraint is a knapsack type constraint. We are aware of two solution attempts in fairly different contexts. Gensch (1978) faces it for optimizing the visits of specialized traveling salesmen employed by a steel firm and propose a Branch and Bound procedure, while Göthe-Lundgren *et al.* (1995) solve it in the context of a cost allocation problem and adopt a heuristic approach. Both solution approaches use sophisticated concepts issued from Lagrangian relaxation theory. These two procedures are presented in the next subsections.

Let us also point out, after Mittenthal and Noon (1992), that the KCPTP includes the STSP as a particular case, when travel costs are set to zero in the objective function, but are considered in the knapsack constraint.

#### B.1 Solving the KCPTP using operator theory

In Gensch's (1978) context, the problem concerns specialists scheduling. Every one of these specialists wants to select some of his customers depending on their sale potentials and schedule a tour so as to minimize travel costs minus profit in an allowed time. In his Branch and Bound solution procedure, Gensch does not focus on the upper bounding scheme and just proposes an adaptation of the TSP nearest neighbor heuristic procedure with vertices selected depending on a profit-to-time ratio. Searching for lower bounds, he relaxes subtour elimination constraints to obtain a time constrained assignment problem. As he wants tight lower bounds, at every node of the search tree he computes an optimal integer solution of this problem. To this end, he relaxes the time constraint in a Lagrangian fashion and solves the Lagrangian multiplier problem with a bisection method, but using the operator theory developed by Srinivasan and Thompson (1972a and 1972b) near the end to catch the optimal integer solution. This Branch and Bound procedure permits to solve instances involving 30 vertices in times varying from 1 second to a few minutes depending on the value of the time limit, on a UNIVAC 1106 system. As expected, computation times are longer for intermediate values, when both decision problems (selection and ordering of the vertices) are nontrivial.

#### B.2 Solving the KCPTP using Lagrangian decomposition

For Göthe-Lundgren *et al.* (1995), the problem consists in allocating the cost of an optimal route configuration to a VRP, fairly, among the customers. Their aim is to compute a lower bound and to deduce a heuristic procedure. To this end, they propose a linear integer programming formulation embedding the knapsack constraint into the subtour elimination constraints. The subtour elimination constraints are then relaxed and a Lagrangian decomposition approach is used, which creates separability by duplicating some variables. Here, $y_i$ variables are duplicated into new variables $z_i$ and matching constraints $y_i = z_i$ (for every $v_i \in V$) are introduced and relaxed in a Lagrangian fashion. For fixed Lagrangian multipliers two separate problems appear, an assignment problem involving $x_i$ and $z_i$ variables and a 0/1 knapsack problem in $y_i$ variables. The difference between the optimal solutions of these two problems gives a lower bound. At each iteration, the assignment problem is solved and subtour elimination constraints violated by the solution are integrated in a Lagrangian fashion for the next iterations. The knapsack problem is also solved and the set of vertices specified by its solution is used to obtain an upper bound with a TSP heuristic. The procedure stops after a high number of iterations or a specified quality of the lower bound compared with the upper bound. Computational results show that the lower bounding method only slightly outperforms a constrained assignment problem based bound (value of the linear relaxation of the problem without considering subtour elimination constraints) at the expense of long computational times. However, both in symmetric and asymmetric cases, the effectiveness of the bounding scheme is undeniable, since the gap between the constrained assignment problem based bound and the optimal solution is also small. Most of the instances, involving up to 100 customers, are computed in less than 100 seconds on a SUN SPARC station ELC. One hard instance requires almost 20 minutes.

### *C. The PCTSP*

The PCTSP involves a second resource on vertices, say prize. The objective is to find a circuit in graph $G$ which minimizes travel costs minus profit and whose collected prize is not smaller than $w_{min}$.

The PCTSP is introduced by Balas and Martin (1985) as a model for scheduling the daily operations of a steel rolling mill (considering a depot or not). This application is detailed below. Fischetti and Toth (1988) notice that this problem also arises when a factory needs a given amount of product, which can be provided by a set of suppliers with given amounts and costs. In a chemical production context, Pekny *et al.* (1990) try to determine schedules for a number of jobs that are candidates for processing over a given time horizon through a system where costs depend only on the consecutive jobs in the production sequence. Such systems could be flowshop with the zero wait processing condition or single machine scheduling with setup times for example. Pekny and Miller (1990) explain that the time horizon limitation might arise when products are shipped via rail or water for example. A cost is incurred if a job is not produced. A benefit is associated to each job completed. Several versions are considered but the problem is solved when the objective is to minimize costs while attaining some level of benefits.

The PCSTSP has soon raised the interest of many researchers. Balas (1989 and 1995) focuses heavily on the structural properties of the PCTSP. He proposes polytope dimension results, many families of facets and valid inequalities (see section II-C). Kubo and Kasugai (1990) describe bounding procedures able to implicitly identify some violated types of valid inequalities, so as to set positive multipliers in a Lagrangian approach or to be used in a Branch and Cut method. Most of the other studies concentrate on finding effective relaxations for Branch and Bound or heuristic procedures. These attempts are described in the next subsections.

Even so, some other studies have arisen for the PCTSP. Balas (1999) introduces ordering constraints for which the PCTSP becomes polynomially solvable. In the same way, Kabadi and Punnen (1996) extend results on polynomially solvable cases of the TSP to the PCTSP. Gueguen (1999) develops a column generation solution procedure for the so-called Prize-collecting VRPTW, involving several vehicles, time windows, service times and capacity constraints. The author repeats the process of a previous column generation solution procedure dedicated to the SVRPTW (analogous extension of the STSP) with only slight modifications. The reader is referred to Feillet *et al.* (2000a) for a detailed description of this procedure in the case of the STSP. The quality of results is comparable in both contexts.

Finally, let us also point out that several procedures dedicated to other TSPs with Profits can be adapted to the PCTSP. One can mention the approximation algorithm from Awerbuch *et al.* (1998) (see section III) or the Göthe-Lundgren *et al.* (1995) heuristic algorithm (see section V-B). On the other hand, a lot of results are assessed on instances with profits set to zero (e.g., Fischetti and Toth 1988 or Dell'Amico *et al.* 1998), which correspond to Quota TSP instances.

### C.1 The steel rolling mill application

The context of steel rolling mills gives rise to complex production scheduling problems and many attempts to solve them. Balas and Martin (1985) develop a software package for scheduling the operations of a steel rolling mill that contains several non described heuristic procedures. At the same time, they introduce the PCTSP, whose interpretation follows, as a simplified and refined version of the kind of problems they face. A rolling mill produces steel sheet from slabs by hot or cold rolling. Schedulers have to choose from an inventory a collection of slabs satisfying a lower bound on total prize and order it so as to minimize some function of the sequence. In reality, schedulers face an enlarged problem involving other resources and complex constraints. Lopez *et al.* (1998) propose a detailed description of the industrial context and describe a fast and effective heuristic based on tabu search addressing it in its whole complexity. The authors also review the related literature. In particular, they describe a study by Cowling (1995) who also proposes a heuristic based on tabu search for a problem which might be seen as a Prize-collecting Vehicle Routing Problem with side constraints. This problem concerns the selection of slabs to be processed from piles in the slab yard so as to determine a set of most desirable feasible sequences of slabs.

### C.2 Bounding schemes for the PCTSP

Many bounding schemes exist for the PCTSP. These schemes are detailed right now, the bold annotations are used for naming and appear as references in the tables ending this section.

1. As for the TSP, a natural way to derive a lower bound is to use the assignment problem substructure of the PCTSP. Fischetti and Toth (1988) and Pekny *et al.* (1990) propose this simple bounding scheme.

- **LP bound.** the subtour elimination constraints are relaxed, which induces a prize-constrained assignment problem. The bound is the value of its linear relaxation. To be more effective this relaxation is computed by the way of a Lagrangian relaxation of the prize constraint. The Lagrangian multiplier problem is solved with a subgradient technique.

Besides its simplicity, this bound is very efficient. As for the TSP, it is more effective for asymmetric instances or when the number of vertices increases.

2. Dell'Amico *et al.* (1995) decide to use the PTP substructure of the PCTSP and develop a bounding scheme based on a Lagrangian relaxation of the prize constraint.

- **DMV.** Prize constraint is relaxed, which induces a PTP instance. This instance is transformed into an Asymmetric TSP instance (see section IV). The bound is its optimal solution. The Asymmetric TSP instance is solved with any efficient existing algorithm.

DMV procedure is only assessed on asymmetric instances. It proves to be quite effective but not very fast. Running times are about 5 times longer than LP bound. This drawback is less detrimental for genuine PCTSP instances (i.e., with positive profits), for which computations are faster, but, unfortunately, no results show the effectiveness of the bound in this last case.

3. Fischetti and Toth (1988) study two other kinds of bounds. The first one is based on the arborescence problem substructure. The second one is based on a disjonction scheme.

- **FT2.** A first approach is to introduce a dummy vertex $v_0$ (see section II) and to relax constraints (2). Authors note that this relaxation authorizes branches in the subtree derived from vertex $v_0$. Thus, it appears quite ineffective and they decide to embed the relaxed constraints in the objective function in a Lagrangian fashion, as well as the prize constraint in order to obtain a shortest spanning 1-arborescence problem. This last problem is solved with a polynomial algorithm while the Lagrangian multiplier problem is solved with a standard subgradient technique.
- **FT3.** The authors use the following disjunction: A feasible solution either visits a distant vertex or is restricted to closer vertices. Thus, solution values either exceed the visitation cost of a distant vertex or the LP bound of the restricted instance.

FT2 appears less effective than the LP bound for asymmetric instances, but it manages better for symmetric instances. The price to pay is much longer computing times. Although it is often very poor, FT3 is effective when the solution involves few vertices.

4. Actually, the main attempt of Fischetti and Toth (1988) is to assess the effectiveness of using these bounds in a so-called additive approach. This approach amounts to finding new bounds in a sequence of increasing lower bounds, each iteration resulting in a bound and a residual instance used for the next iteration. This approach is evaluated with different combinations.

- **FT4.** The assignment based bound is applied in sequence, with a shrinking of the graph at each iteration, necessary to make possible an improvement of the result.
- **FT5.** The assignment based bound and the disjunction based bound are applied in sequence.

These two bounding schemes prove to be very effective for symmetric instances. For asymmetric instances, they only slightly outperform the LP bound, but computing times are almost as short.

5. Göthe-Lundgren *et al.* (1995) also tackle the PCTSP with the bounding scheme they initially developed for the KCPTP.

- **GMV.** The bound is computed through Lagrangian decomposition. (see section V-B for details).

The procedure behaves well, but running times are far longer than for other procedures.

Tables I and II report the effectiveness of the different bounding schemes for symmetric and asymmetric instances. In these tables, $\alpha$ is the ratio of prize to be collected over the total prize value for all vertices (instances with $\alpha$ sets to 1 are TSP instances) and $\sharp$ vertices is the number of vertices in the graph. The bound and solution values reported in the tables are normalized ones: The original values are divided by the value of the linear relaxation when subtour elimination constraints are ignored (LP bound).

These tables have to be read carefully. All results are computed on instances constructed using the same rules, but these instances are not exactly identical. Results involving different instances are separated with vertical lines. In spite of this weakness, we think that these results permit to clarify the behaviour of the bounding schemes depending on the parameters and to have an idea of their comparative effectiveness.

### C.3 Branch and Bound procedures for the PCTSP

A first Branch and Bound procedure is proposed by Fischetti and Toth (1988), after having thoroughly studied bounding schemes. The two bounding schemes **FT4** and **FT5** using the additive approach described previously are embedded in the procedure. Asymmetric and symmetric instances with up to, respectively, 100 and 40 vertices are solved in a few minutes on a Digital VAX 11/780.

Pekny *et al.* (1990) develop a parallel Branch and Bound algorithm. Lower bounds are based on the LP bound described above. Upper bounds are computed using an extension of Karp's (1984) patching algorithm, combining the subtours of the relaxed solution derived from the lower bound and eventually incorporating non visited vertices to form a feasible tour. Computational results show this parallel Branch and Bound procedure to be effective on asymmetric instances with up to 200 vertices.

### C.4 Heuristic procedures for the PCTSP

Dell'Amico *et al.* (1998) present two heuristic procedures for the PCTSP. In the first heuristic procedure, the lower bounding procedure described previously is first executed and the computed cycle is used to obtain a feasible solution with a profit-to-cost driven insertion procedure. This solution is improved with two procedures, Extension and Collapse, applied iteratively until no further improvement is obtained. The Extension procedure determines the average profit-to-cost ratio of an insertion and applies the insertion procedure as long as insertions are over this average ratio. The Collapse procedure determines all the maximal (inclusionwise) infeasible subcycles of the current solution, adds a vertex at minimum cost to each of them and selects the cheapest one. The second heuristic uses the same components, but in a different order. In particular, the Extension and Collapse procedures are applied during the computation of the lower bound. This second heuristic proves to be more effective. Results confirm that the higher the penalties are, the better the results are, since the prize constraint is more naturally satisfied in this case and the lower bound often corresponds to a feasible cycle.

## VI. Other extensions for the PTP

Besides the straight previous extensions of the PTP, one can find several more elusive problems which derive from the PTP. A recurrent framework is the problems faced by carriers for the transportation of freight. These applications are described at the end of this section.

Actually, the PTP is also exactly the problem of finding a minimum cost elementary cycle or circuit visiting the depot in a graph with negative costs. When the depot condition is dropped, this problem is encountered in undirected graphs under the names of Weighted Girth Problem, Circuit Problem or Subtour Problem. In this situation, one can find many structural studies (see section II-C). Kubo and Kasugai (1992) also propose several mixed integer programming formulations and compare the bounds derived from continuous relaxation of these formulations. Bauer (1997) explains that the study of this problem is motivated by the fact that it appears with slight variations as a subproblem in a column generation approach to vehicle routing problems. Indeed, Bauer *et al.* (1998) introduce this related problem under the name of Knapsack Constrained Circuit Problem. It consists in finding an elementary circuit which minimizes travel costs with an arbitrary cost matrix and a knapsack type constraint for the new resource. This problem is encountered to model the pricing problem in Branch and Price algorithms for the VRP (the knapsack constraint stands for capacity restrictions). Bauer *et al.* (1998) propose a Branch and Cut algorithm in the case of unit consumption of the resource (Cardinality Constrained Circuit Problem).

With some additional constraints, but with the depot condition, Gueguen *et al.* (2000) introduce the Elementary Shortest Path Problem with Resource Constraints, which is also used in Branch and Price algorithms, but for time-constrained VRP. They solve it using dynamic programming.

*Carriers transportation problems*

Bookbinder and Sural (1999) consider the situation of a carrier who makes deliveries and also has a number of optional backhauls. The objective is to find the least cost tour by selecting the most profitable backhauls. For this purpose, they study the related polyhedron, tightening the subtour elimination constraints and developing valid inequalities.

Diaby and Ramesh (1995) introduce a problem they call the Distribution Problem with Carrier Service that arises

TABLE I
BOUNDING SCHEMES FOR THE SYMMETRIC PCTSP.

| | | $p_i = 0$ (Quota TSP) | | | | | |
|---|---|---|---|---|---|---|---|
| | | Fischetti and Toth (1988) | | | | Göthe-Lundgren *et al.* (1995) | |
| $\alpha$ | ♯ vertices | FT2 | FT4 | FT5 | Sol. value | GMV | Upper bound |
| 0.2 | 20 | 1.101 | 2.133 | 2.250 | 2.969 | 1.381 | 2.302 |
| | 40 | 1.181 | 2.859 | 2.523 | 3.589 | 1.544 | 5.515 |
| | 60 | 1.250 | 4.267 | 3.665 | | 1.244 | 5.655 |
| | 80 | 1.136 | 2.669 | 2.186 | | 1.258 | 6.622 |
| | 100 | 1.253 | 2.498 | 1.893 | | 1.203 | 5.469 |
| 0.5 | 20 | 1.148 | 1.772 | 1.707 | 2.245 | 1.234 | 2.092 |
| | 40 | 1.285 | 2.010 | 1.576 | 2.291 | 1.257 | 2.675 |
| | 60 | 1.343 | 2.119 | 1.743 | | 1.252 | 2.846 |
| | 80 | 1.263 | 1.948 | 1.462 | | 1.241 | 2.698 |
| | 100 | 1.184 | 1.788 | 1.301 | | 1.221 | 2.655 |
| 0.8 | 20 | 1.150 | 1.433 | 1.322 | 1.697 | 1.192 | 1.633 |
| | 40 | 1.209 | 1.446 | 1.288 | 1.597 | 1.274 | 1.795 |
| | 60 | 1.280 | 1.510 | 1.274 | | 1.269 | 1.813 |
| | 80 | 1.239 | 1.470 | 1.191 | | 1.302 | 1.860 |
| | 100 | 1.255 | 1.480 | 1.148 | | 1.275 | 1.848 |
| 1 | 20 | 1.039 | 1.188 | 1.116 | 1.277 | 1.372 | 1.420 |
| | 40 | 1.204 | 1.190 | 1.107 | 1.278 | 1.212 | 1.263 |
| | 60 | 1.199 | 1.167 | 1.084 | | 1.224 | 1.303 |
| | 80 | 1.211 | 1.184 | 1.082 | | 1.248 | 1.314 |
| | 100 | 1.238 | 1.204 | 1.065 | | 1.206 | 1.270 |

TABLE II
BOUNDING SCHEMES FOR THE ASYMMETRIC PCTSP.

| | | $p_i = 0$ (Quota TSP) | | | | | | | $1 \leq p_i \leq 100$ | $1 \leq p_i \leq 10000$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | Fischetti and Toth (1988) | | | | Göthe-Lundgren *et al.* (1995) | | | |
| $\alpha$ | ♯ vertices | DMV | FT2 | FT4 | FT5 | Sol. value | GMV | Upper bound | DMV | DMV |
| 0.2 | 20 | 1.050 | 0.949 | 1.137 | 1.142 | 1.360 | 1.067 | 1.351 | 1.047 | 1.059 |
| | 40 | 1.037 | 0.985 | 1.125 | 1.109 | 1.300 | 1.106 | 1.302 | 1.023 | 1.026 |
| | 60 | 1.054 | 0.899 | 1.007 | 1.020 | 1.140 | 1.011 | 1.163 | 1.014 | 1.019 |
| | 80 | 1.060 | 0.853 | 1.021 | 1.018 | 1.102 | 1.010 | 1.203 | 1.013 | 1.012 |
| | 100 | 1.056 | 0.850 | 1.015 | 1.009 | 1.073 | 1.008 | 1.327 | 1.007 | 1.011 |
| 0.5 | 20 | 1.148 | 1.022 | 1.097 | 1.076 | 1.303 | 1.114 | 1.393 | 1.069 | 1.063 |
| | 40 | 1.047 | 0.983 | 1.053 | 1.034 | 1.122 | 1.033 | 1.216 | 1.019 | 1.028 |
| | 60 | 1.055 | 0.954 | 1.006 | 1.001 | 1.059 | 1.021 | 1.278 | 1.022 | 1.023 |
| | 80 | 1.031 | 0.928 | 1.005 | 1.004 | 1.032 | 1.000 | 1.112 | 1.012 | 1.017 |
| | 100 | 1.031 | 0.942 | 1.006 | 1.001 | 1.040 | 1.001 | 1.100 | 1.010 | 1.008 |
| 0.8 | 20 | 1.058 | 1.027 | 1.049 | 1.049 | 1.153 | 1.062 | 1.194 | 1.028 | 1.059 |
| | 40 | 1.041 | 0.987 | 1.013 | 1.011 | 1.057 | 1.013 | 1.114 | 1.027 | 1.026 |
| | 60 | 1.032 | 0.977 | 1.004 | 1.001 | 1.036 | 1.007 | 1.083 | 1.014 | 1.019 |
| | 80 | 1.022 | 0.982 | 1.005 | 1.001 | 1.028 | 1.005 | 1.169 | 1.013 | 1.012 |
| | 100 | 1.015 | 0.977 | 1.003 | 1.001 | 1.015 | 1.001 | 1.113 | 1.007 | 1.011 |
| 1 | 20 | | 0.938 | 1.002 | 1.002 | 1.057 | 1.046 | 1.079 | | |
| | 40 | | 0.941 | 1.000 | 1.000 | 1.017 | 1.017 | 1.060 | | |
| | 60 | | 0.942 | 1.000 | 1.000 | 1.012 | 1.008 | 1.040 | | |
| | 80 | | 0.972 | 1.001 | 1.001 | 1.009 | 1.007 | 1.028 | | |
| | 100 | | 0.924 | 1.000 | 1.000 | 1.006 | 1.007 | 1.039 | | |

in the distribution of commodities. A single vehicle delivers customers along a tour starting and ending at the depot, while respecting capacity and time limit constraints. A given cost is paid for non visited customers as they are served by an outside carrier. The problem is an extension of the PTP with an additional time limit and a capacity constraint. The problem is solved using a Branch and Bound strategy. Upper bounds are obtained using an adaptation of the heuristic developed by Ramesh and Brown (1991) for the STSP. The lower bounding scheme relies on the linear programming relaxation of the problem. To this end, the authors study structural properties of the polytope. They solve the assignment problem based relaxation and improve it by incorporating violated valid inequalities. The procedure manages to solve instances with up to 200 vertices in a few minutes on an IBM 3081/GX. This approach is used in the context of an industrial chemical company. Besides the previous facts, it involves a dynamic component. A trip is initiated each day, but trips are completed in two working days. A rolling horizon approach is used to take into account this characteristic. The authors report impressive savings illustrating the operational efficiency of the model compared to the current system used in the company.

Another transportation problem is addressed by Feillet *et al.* (2000b) and concerns the tactical planning of freight transportation between plants in the car industry. Freight movements are planned in advance, but two transportation methods with different costs can be used to realize them.

The more expensive method consists in one-way journeys, while the other consists in circuits. The authors start the optimization with the inferior solution of only using one-way journeys. Revising this initial choice for some freight movements then amounts to solving a PTP involving several vehicles and no depot. This problem is tackled with a column generation approach and exact solutions to real-life instances with up to 20 plants are obtained using a Branch and Price strategy in a few seconds.

## VII. The STSP

The objective of the STSP is to find an elementary circuit including the depot in graph $G$ that maximizes collected profit, with travel costs not exceeding a given bound. Besides the previously considered situations (traveling salesman problems, job scheduling...), the problem has been addressed for an inventory routing problem (Golden *et al.* 1984) or in an orienteering competition context (Tsiligirides 1984). Many solution procedures have been proposed, extending the major fields of investigation for solving the TSP. In particular, many construction and improvement based heuristics have been developed (e.g., Golden *et al.* 1988, Ramesh and Brown 1991, Chao *et al.* 1996). Exact solution procedures have also been addressed, as Branch and Bound procedures (e.g., Laporte and Martello 1990, Ramesh *et al.* 1992) or Branch and Cut procedures (Fischetti *et al.* 1998, Gendreau *et al.* 1998a). At last, the problem structure and its special features (especially the possibility of dropping a vertex) have favored the development of metaheuristic approaches (e.g., Tabu search, Gendreau *et al.* 1998b).

The multi-vehicle version has seen less consideration, but far more than for the other TSPs with Profits. A lot of practical situations have been considered: Aerial reconnaissance (Moser 1990), underway replenishment of ships (Wu 1992), athlete recruiting (Butt and Cavalier 1994), fisheries patrol (Millar and Kiragu 1997) and tourist route determination (Deitch and Ladany 2000). Among the developed procedures emerge two column generation algorithms (Butt and Ryan 1999, Gueguen 1999).

All these research works are deeply detailed in Feillet *et al.* (2000a).

## VIII. Conclusion

In this paper, we have proposed a classification and a commented survey regarding a class of problems we named TSPs with Profits. This class includes monocriterion versions of a bicriteria extension of the TSP, which consists in selecting a set of customers to visit according to a profit, while minimizing travel costs. Many attempts have been made to solve these problems but the lack of a clearly defined classification appears strongly. Our first goal was to emphasize the differences and common features of the different versions, and to clarify the connections with related problems. These relations are summarized in figure 3. Dotted lines indicate relations when data are specific.

TSPs with Profits occur in a wide variety of situations, including realistic traveling salesmen problems, job scheduling, freight transportation or, indirectly, as a subproblem in solution approaches dedicated to other problems. Our second goal was to confirm the necessity to tackle this class of problem. Tables III and IV summarize all these applications.

Our third goal was to bring together all the attempts dedicated to one of these problems. Researchers have considered them at different levels. Indeed, this class is closely related to the TSP but with a very specific structure. This new structure has been the core subject of several papers studying structural properties or bounding schemes based on linear programming formulations (e.g., Fischetti and Toth 1988, Balas 1989, Kubo and Kasugai 1992, Dell'Amico *et al.* 1995). At a completely different level, papers take it into account for including novel features in heuristic procedures otherwise widely adapted from classical TSP heuristics such as insertion procedures (e.g., Mittenthal and Noon 1992). Other researchers have been more interested in assessing more recent or technical methods for this kind of problems (e.g., Pekny *et al.* 1990). We believe that all of these approaches are praiseworthy since one needs simple heuristic algorithms, easy to implement and fast, as well as more advanced ones, complex and more time-consuming, but more effective. These approaches are summarized in table V.

We have to point out that, although TSPs with Profits have received a lot of attention, very few attempts have been made to tackle their multi-vehicle extension. Yet, this class of problems is something of an intermediary problem between TSP and VRP, in so far as the vehicle visits only a subset of vertices. Besides, many of its applications could as well be extended to the multi-vehicle situation. It would certainly be interesting to develop solution procedures dedicated to this extension.

Even if they are not straight extensions of TSPs with Profits, we have to mention the Traveling Purchaser Problem and the Generalized Traveling Salesman Problem, which are two other extensions of the TSP in which customers have to be selected. The Traveling Purchaser Problem involves a set of commodities. Each vertex (standing for a market) holds commodities, each with its own associated purchase cost. The objective is to find a tour (including the depot) that enables to purchase all the commodities so as to minimize the total travel and purchase cost. Readers are referred to Voß (1996), Singh and van Oudheusden (1997), Pearn and Chien (1998) or Boctor *et al.* (2001) to find more information. In the Generalized Traveling Salesman Problem, vertices are components of clusters and the aim is to find a minimum cost circuit visiting at least one vertex of each cluster. Recent investigations are found in Fischetti *et al.* (1997) or Renaud and Boctor (1998). The multi-vehicle version of this last problem has also been studied in Ghiani and Improta (2000). Finally, far removed extensions of the TSP for which customers are selected can be found under such names as the Shortest Covering Path Problem (Current *et al.* 1994), the Covering Tour Problem (Gendreau *et al.* 1997) or the Traveling Circus Problem (ReVelle and Laporte 1993). More details

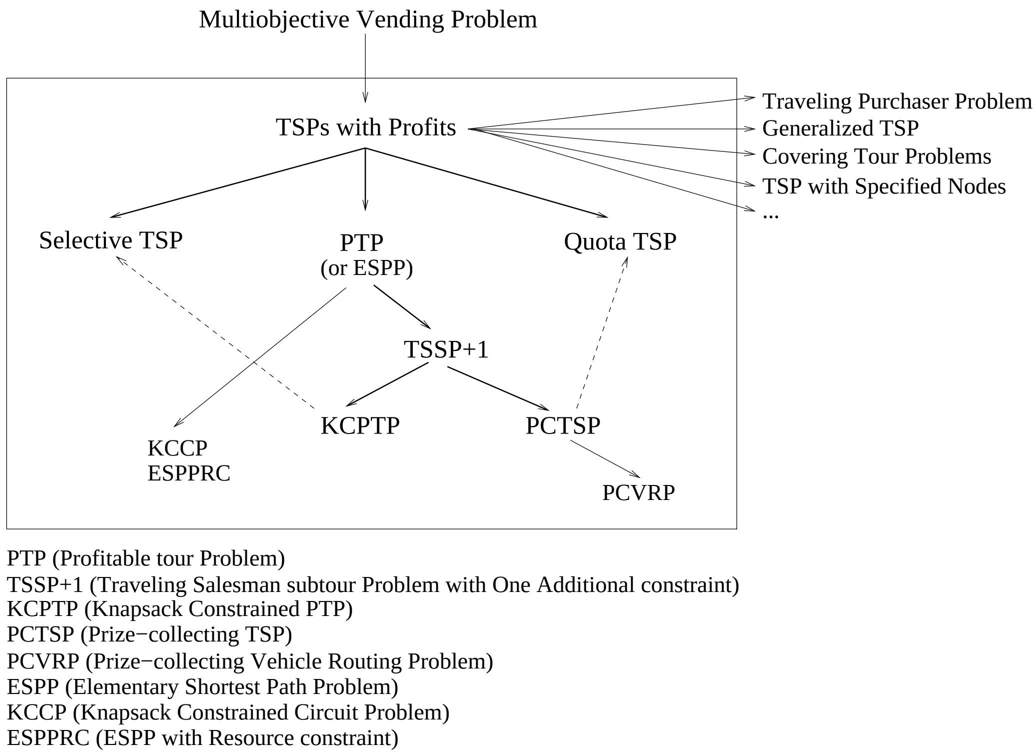


Fig. 3. TSPs with Profits and related problems

TABLE III

USING TSPS WITH PROFITS FOR SOLVING INDUSTRIAL APPLICATIONS.

| Application | TSP with Profits | Reference |
|---|---|---|
| Traveling salesmen scheduling | KCPTP | Gensch (1978) |
| Steel rolling mill scheduling | PCTSP | Balas and Martin (1985) |
| | PCTSP + constraints | Lopez *et al.* (1998) |
| | PCVRP + constraints | Cowling (1995) |
| Scheduling with an aggregate deadline | PCTSP | Pekny *et al.* (1990) |
| | PCTSP | Pekny and Miller (1990) |
| Carriers transportation problems | PTP + constraints | Bookbinder and Sural (1999) |
| | PTP + constraints | Diaby and Ramesh (1995) |
| | PTP + constraints | Feillet *et al.* (2000b) |

about these can be found in Labbé *et al.* (1998). Finally, we have to mention a slightly different but related problem, where the aim is to determine the shortest circuit (or path) in a graph where some specified vertices have to be visited exactly once while the others are visited at most once (see Laporte *et al.* 1984 and Volgenant and Jonker 1987).

It is important to recall that the Selective Traveling Salesman is not covered in this survey, in spite of having received a lot more attention by the research community. This problem is the object of a complete companion survey to this one (Feillet *et al.* 2000a).

By way of conclusion, our opinion is that in the field of traveling salesman problems, the condition imposing the visit of every customer is not always relevant. However, people are often inclined to focus on a low level of decision and to tackle the optimization of vehicles visiting the complete set of vertices, eventually after having restricted it. Yet results on TSPs with Profits show that it is not necessarily more difficult to merge vertex selection and travel cost optimization. We expect that more and more situations will be considered this way and to see many more practical situations modeled using this framework in the future.

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TABLE IV

Using TSPs with Profits as subproblems.

| Problem | TSP with Profits | Reference |
|---|---|---|
| m-Cost ATSP | PTP | Helmberg (1999) |
| VRP | TSSP+1 | Noon *et al.* (1994) |
| | KCCP | Bauer *et al.* (1998) |
| VRP with Time Windows | ESPPRC | Gueguen *et al.* (2000) |
| Cost allocation problem | KCPTP | Göthe-Lundgren *et al.* (1995) |

TABLE V

Approaches for solving TSPs with Profits.

| Procedure | TSP with Profits | Reference |
|---|---|---|
| Approximation algorithm | Quota TSP, PCTSP | Awerbuch *et al.* (1998) |
| | PTP | Bienstock *et al.* (1993) |
| | PTP | Goemans and Williamson (1995) |
| Classical heuristic | TSSP+1 (KCPTP, PCTSP) | Mittenthal and Noon (1992) |
| | Quota TSP, PCTSP | Dell'Amico *et al.* (1998) |
| Lagrangian heuristic | PTP, Quota TSP | Göthe-Lundgren *et al.* (1995) |
| | Quota TSP, PCTSP | Dell'Amico *et al.* (1998) |
| Tabu search | PCTSP + constraints | Lopez *et al.* (1998) |
| | PCVRP + constraints | Cowling (1995) |
| Dynamic Programming | ESPPRC | Gueguen *et al.* (2000) |
| Branch and Bound | KCPTP | Gensch (1978) |
| | PCTSP | Fischetti and Toth (1988) |
| | PTP + constrainsts | Diaby and Ramesh (1995) |
| Parallel Branch and Bound | PCTSP | Pekny *et al.* (1990) |
| Branch and Cut | KCCP | Bauer *et al.* (1998) |
| Column generation | PCVRP with Time Windows | Gueguen (1999) |
| | PTP + constraints | Feillet *et al.* (2000b) |


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