APPENDIX: DERIVATIONS

This appendix provides details of derivations referred to throughout the text. The derivations are developed here so that the main topics within the chapters are not interrupted by the sometimes lengthy mathematical developments.

DERIVATION OF MAGNETIC DIPOLE EQUATIONS

In this section, a derivation is provided of the basic equations describing the magnetic field produced by a magnetic dipole. The geometry is shown in Figure A.1 and is identical to the geometry of Figure 1.3 for a geocentric axial dipole. The derivation is developed by using spherical coordinates: r, θ, and ϕ. An additional polar angle, p, is the *colatitude* and is defined as $\pi - \theta$. After each quantity is derived in spherical coordinates, the resulting equation is altered to provide the results in the convenient forms (e.g., horizontal component, H_h) that are usually encountered in paleomagnetism.

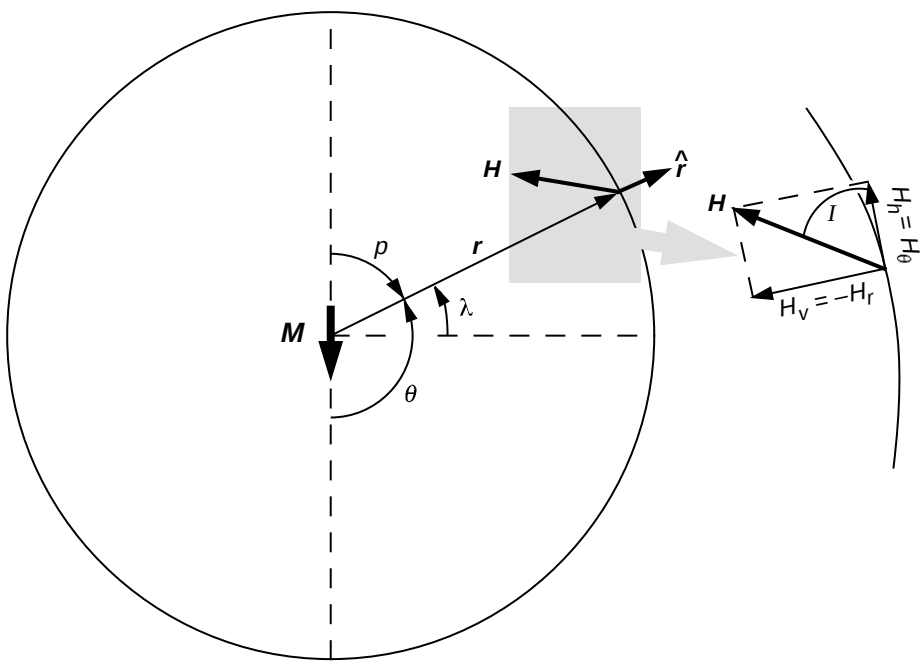

Figure A.1 Geocentric axial dipole. The large arrow is the magnetic dipole moment, ***M***; θ is the polar angle from the positive pole of the magnetic dipole; p is the magnetic colatitude; λ is the geographic latitude; ***r*** is the radial distance from the magnetic dipole; ***H*** is the magnetic field produced by the magnetic dipole; $\hat{r}$ is the unit vector in the direction of ***r***. The inset figure in the upper right corner is a magnified version of the stippled region. Inclination, I, is the vertical angle (dip) between the horizontal and ***H***. The magnetic field vector ***H*** can be broken into (1) vertical component, $H_v = -H_r$, and (2) horizontal component, $H_h = H_\theta$.

The starting point is the scalar magnetic potential of a magnetic dipole:

$$V = \frac{\mathbf{M} \cdot \hat{\mathbf{r}}}{r^2} = \frac{M \cos\theta}{r^2} \tag{A.1}$$

The magnetic field, ***H***, is derived from the scalar magnetic potential by taking the gradient of the potential:

$$\mathbf{H} = -\nabla \mathrm{V} = -\left(\frac{\partial}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta} \right) \left(\frac{M \cos\theta}{r^2} \right) \tag{A.2}$$

Separating the differentials yields

$$\mathbf{H} = -\frac{\partial}{\partial r} \left(\frac{M \cos\theta}{r^2} \right) \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{M \cos\theta}{r^2} \right) \hat{\theta} \tag{A.3}$$

Performing the required differentiations leads to

$$\mathbf{H} = \frac{2M \cos\theta}{r^3} \hat{\mathbf{r}} + \frac{M \sin\theta}{r^3} \hat{\theta} = H_r \hat{\mathbf{r}} + H_\theta \hat{\theta} \tag{A.4}$$

The horizontal component, H_h, of ***H*** is then given by

$$H_h = H_\theta = \frac{M \sin\theta}{r^3} = \frac{M \sin(\pi - \theta)}{r^3} = \frac{M \sin p}{\mathrm{r}^3} \tag{A.5}$$

To get this expression in terms of geographic latitude, λ, substitute

$$p = \frac{\pi}{2} - \lambda \tag{A.6}$$

to yield

$$H_h = \frac{M \cos\lambda}{r^3} \tag{A.7}$$

This is Equation (1.12) in Chapter 1.

Now returning to Equation (A.4), the vertical component, H_v, of ***H*** is

$$H_v = -H_r = -\frac{2M \cos\theta}{r^3} = \frac{2M \cos p}{r^3} \tag{A.8}$$

Again using Equation (A.6), H_v in terms of geographic latitude, λ, is

$$H_v = \frac{2M \sin\lambda}{r^3} \tag{A.9}$$

This is Equation (1.13).

The inclination, I, can be determined by

$$\tan I = \frac{H_v}{H_h} = \left(\frac{2M \cos p}{r^3} \right) \left(\frac{r^3}{M \sin p} \right) = 2 \cot p \tag{A.10}$$

Using Equation (A.6), the inclination is given as a function of geographic latitude by

$$\tan I = 2 \tan \lambda \tag{A.11}$$

This is Equation (1.15), “the dipole equation.”

For the total intensity, H, of the magnetic field, we find

$$H = \sqrt{H_h^2 + H_v^2} = \frac{M}{r^3}\sqrt{1+3\cos^2 p} = \frac{M}{r^3}\sqrt{1+3\sin^2 \lambda} \tag{A.12}$$

which is Equation (1.14).

ANGLE BETWEEN TWO VECTORS (AND GREAT-CIRCLE DISTANCE BETWEEN TWO GEOGRAPHIC LOCATIONS)

The dot product (scalar product) of two vectors ***A*** and ***B*** is given by

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta \tag{A.13}$$

where A is the length of vector ***A***, B is the length of vector ***B***, and θ is the angle between ***A*** and ***B***.

In terms of the components of the vectors in Cartesian coordinates,

$$\mathbf{A}\cdot\mathbf{B} = A_x B_x + A_y B_y + A_z B_z \tag{A.14}$$

where B_x is the x component of ***B***, etc.

The angle θ can be determined by

$$\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right) \tag{A.15}$$

Now instead of dealing with Cartesian coordinates, express the directions in terms of north, east, and down components on a sphere. For example, a unit vector $\hat{\mathbf{A}}$ can be expressed as

$$\hat{\mathbf{A}} = A_N\hat{\mathbf{N}} + A_E\hat{\mathbf{E}} + A_V\hat{\mathbf{V}} \tag{A.16}$$

where A_N is the north component of $\hat{\mathbf{A}}$, etc.

The unit vector $\hat{\mathbf{A}}$ can be written in terms of its inclination and declination by

$$A_N = \cos I_A \cos D_A,\ A_E = \cos I_A \sin D_A,\ \text{and } A_V = \sin I_A \tag{A.17}$$

where I_A is the inclination of unit vector $\hat{\mathbf{A}}$, etc.

Now the angle between two directions (unit vectors) can be written as

$$\theta = \cos^{-1}\left(\frac{\hat{\mathbf{A}}\cdot\hat{\mathbf{B}}}{\hat{A}\hat{B}}\right) = \cos^{-1}\left(\hat{\mathbf{A}}\cdot\hat{\mathbf{B}}\right) \tag{A.18}$$

In terms of the inclinations and declinations of the two vectors, the angle θ is given by

$$\theta = \cos^{-1}(\cos I_A \cos D_A \cos I_B \cos D_B + \cos I_A \sin D_A \cos I_B \sin D_B + \sin I_A \sin I_B) \tag{A.19}$$

So given the inclinations and declinations of any two vectors, one can use Equation (A.19) to determine the angle between those two directions.

Equation (A.19) also can be used to determine the great-circle distance (angular distance) between any two geographic locations. Instead of viewing directions as being points on a sphere of unit radius, we now use the unit sphere to view geographic locations. Consider two geographic locations, one at latitude λ_a and longitude ϕ_a and another at latitude λ_b and longitude ϕ_b. The great-circle distance from (λ_a, ϕ_a) to (λ_b, ϕ_b) is determined by substituting λ_a for I_A, ϕ_a for D_A, etc. in Equation (A.19). The result is

$$\theta = \cos^{-1}(\cos \lambda_a \cos \phi_a \cos \lambda_b \cos \phi_b + \cos \lambda_a \sin \phi_a \cos \lambda_b \sin \phi_b + \sin \lambda_a \sin \lambda_b) \tag{A.20}$$

An alternative expression for the great-circle distance between two locations is introduced below and is sometimes the more convenient form.

LAW OF SINES AND LAW OF COSINES

Two fundamental relationships of spherical trigonometry can be illustrated by using the spherical triangle *ABC* in Figure A.2, and these relationships will be used often in the derivations to follow. The spherical triangle has corners *A*, *B*, and *C*; and *A*, *B*, and *C* stand for the angles between the sides of the triangle at the respective corners. The distances *a*, *b*, and *c* are angular distances of the sides of the triangle that are opposite the corners *A*, *B*, and *C*, respectively. These angular distances are the angle subtended by a side of the triangle at the center of the sphere (see the inset in Figure A.2).

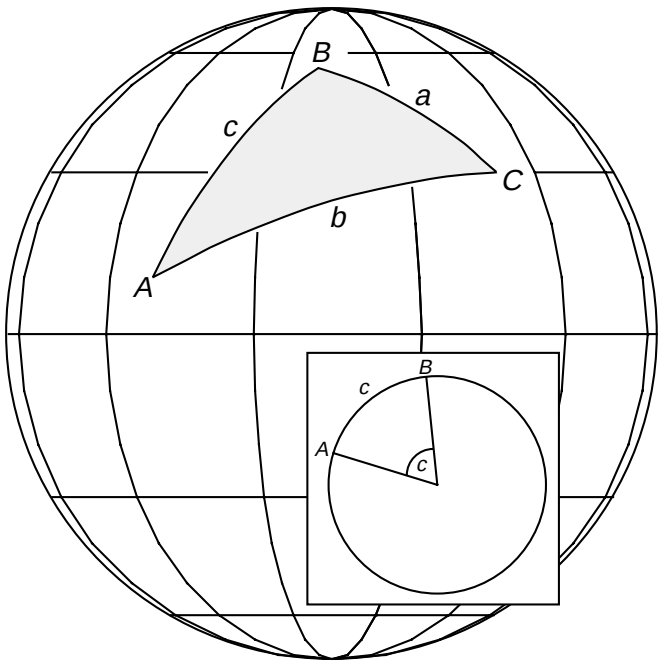

Figure A.2 Spherical triangle with apices at *A*, *B*, and *C* and sides *a*, *b*, and *c*. The inset figure shows the plane containing *A*, *B*, and the center of the sphere; the angular distance *c* is the angle subtended by side *c* at the center of the sphere. The projection (for this and all global projections to follow) is orthographic with the latitude and longitude grid in 30° increments.

The *Law of Cosines* states that

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \tag{A.21}$$

The Law of Cosines can be applied to any side of a spherical triangle by simply rearranging the labels on the sides and at the corners. For example, in the triangle of Figure A.2,

$$\cos b = \cos c \cos a + \sin c \sin a \cos B \tag{A.22}$$

The second relationship is the *Law of Sines*, for which the governing equation is

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \tag{A.23}$$

We will apply the Law of Cosines and the Law of Sines frequently in the coming derivations.

CALCULATION OF A MAGNETIC POLE FROM THE DIRECTION OF THE MAGNETIC FIELD

The trigonometry involved in deriving the expressions for calculating a magnetic pole from a magnetic field direction is shown in Figure A.3a. The site is at geographic latitude λ_s and longitude ϕ_s and the pole is at geographic latitude λ_p and longitude ϕ_p. We form a spherical triangle with apices at (λ_s, ϕ_s), (λ_p, ϕ_p), and the

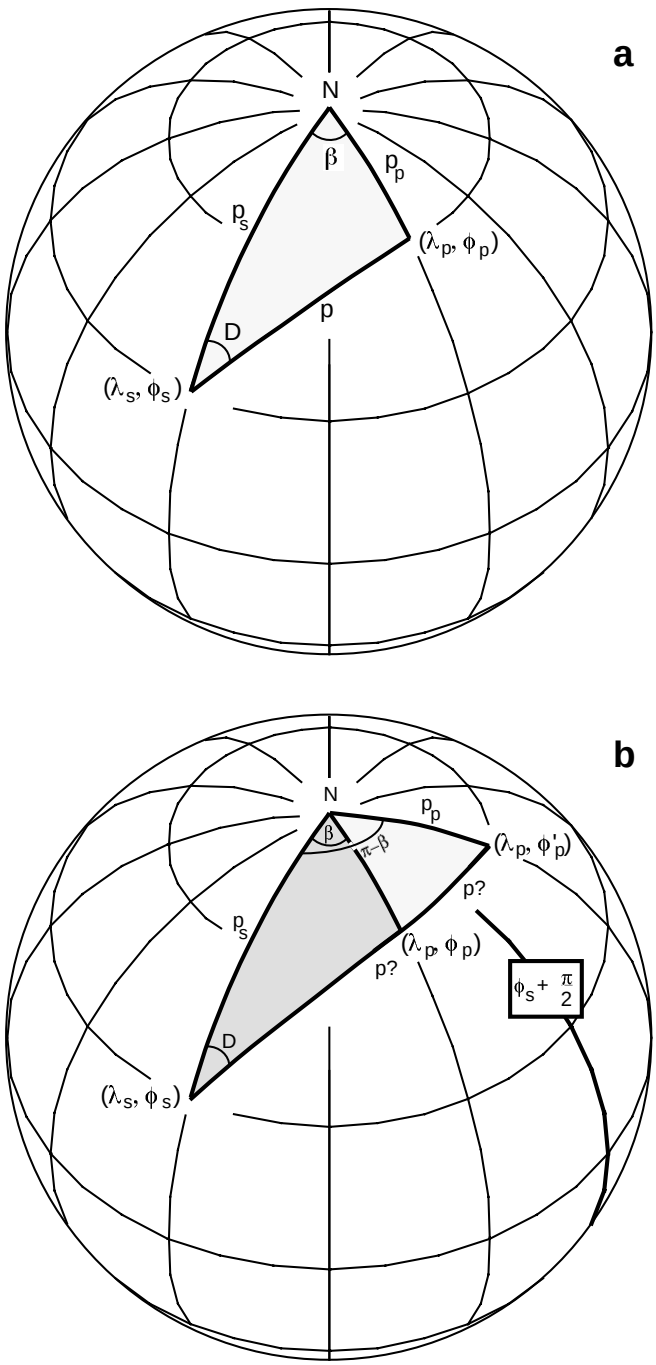

Figure A.3 (**a**) Determination of a magnetic pole from a magnetic field direction. The site is at (λ_s, ϕ_s); the magnetic pole is at (λ_p, ϕ_p); the north geographic pole is at point *N*; the colatitude of the site is p_s; the colatitude of the magnetic pole is p_p; β is the longitudinal difference between the magnetic pole and the site. (**b**) Ambiguity in magnetic pole longitude. The pole may be at either (λ_p, ϕ_p) or (λ_p, ϕ_p'); the longitude at $\phi_s + \pi/2$ is shown by the heavy line.

north geographic pole, *N*. The colatitude (angular distance from the north geographic pole) of the site is p_s, while the colatitude of the magnetic pole is p_p.

The magnetic colatitude, *p*, is the great-circle angular distance of the site from the magnetic pole. This angular distance is determined from the dipole formula (Equation (A.10)):

$$p = \cot^{-1}\left(\frac{\tan I}{2}\right) = \tan^{-1}\left(\frac{2}{\tan I}\right) \tag{A.24}$$

Now we need to find p_p by using the Law of Cosines:

$$\cos p_p = \cos p_s \cos p + \sin p_s \sin p \cos D \tag{A.25}$$

Using the definition of the colatitude,

$$p_p = \frac{\pi}{2} - \lambda_p \quad \text{and} \quad p_s = \frac{\pi}{2} - \lambda_s \tag{A.26}$$

Substituting these expressions for p_p and p_s in Equation (A.25) leads to

$$\cos\left(\frac{\pi}{2} - \lambda_p\right) = \cos\left(\frac{\pi}{2} - \lambda_s\right)\cos p + \sin\left(\frac{\pi}{2} - \lambda_s\right)\sin p \cos D \tag{A.27}$$

Using

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x \quad \text{and} \quad \sin\left(\frac{\pi}{2} - x\right) = \cos x$$

in Equation (A.27) yields

$$\sin\lambda_p = \sin\lambda_s \cos p + \cos\lambda_s \sin p \cos D \tag{A.28}$$

and

$$\lambda_p = \sin^{-1}\left(\sin\lambda_s \cos p + \cos\lambda_s \sin p \cos D\right) \tag{A.29}$$

which is Equation (7.2).

The next step is to determine the angle β, which is the difference in longitude between the pole and the site (Figure A.3a). Applying the Law of Sines to the spherical triangle in Figure A.3 yields

$$\frac{\sin p}{\sin\beta} = \frac{\sin p_p}{\sin D} \tag{A.30}$$

Rearrange Equation (A.30) to give

$$\sin\beta = \frac{\sin D}{\sin p_p}\sin p \tag{A.31}$$

Now substitute $p_p = (\pi / 2) - \lambda_p$ to yield

$$\sin\beta = \frac{\sin D}{\sin\left(\frac{\pi}{2} - \lambda_p\right)}\sin p \tag{A.32}$$

and

$$\sin\beta = \frac{\sin D}{\cos\lambda_p}\sin p \tag{A.33}$$

Now solve for β to give

$$\beta = \sin^{-1}\left(\frac{\sin p \sin D}{\cos\lambda_p}\right) \tag{A.34}$$

which is Equation (7.3).

As given by Equation (A.34), β is limited to the range $-\pi / 2$ to $+\pi / 2$. But this raises an important ambiguity in the derivation. Simply adding b to the site longitude would not allow the pole longitude to differ from the site longitude by more than $\pi / 2$. This ambiguity is shown schematically in Figure A.3b. As viewed from the site at λ_s, ϕ_s, the above expression for β would not allow the pole to be in the longitudinal hemisphere opposite from the site (beyond the longitude shown by the heavy line in Figure A.3b).

The ambiguity is whether the pole longitude is given by (1) $\phi_p = \phi_s + \beta$ (as shown in Figure A.3a) <u>or</u> (2) $\phi_p = \phi_s + (\pi - \beta)$. These two possibilities are shown by the two spherical triangles in Figure A.3b. The

smaller triangle has apices at (λ_s, ϕ_s), (λ_p, ϕ_p), and N; the larger triangle has apices at (λ_s, ϕ_s), (λ_p, ϕ'_p), and N. Because λ_p is the same for either of the two possible poles, p_p is the same angular distance for either triangle. So we must devise a test to determine which of the two possible spherical triangles applies to a particular calculation of a magnetic pole position.

Apply the Law of Cosines to the two triangles in Figure A.3b. For the smaller triangle,

$$\cos p = \cos p_p \cos p_s + \sin p_p \sin p_s \cos\beta \tag{A.35}$$

while for the larger triangle,

$$\cos p = \cos p_p \cos p_s + \sin p_p \sin p_s \cos(\pi - \beta) \tag{A.36}$$

Now substitute

$$p_p = \left(\frac{\pi}{2} - \lambda_p\right),\ p_s = \left(\frac{\pi}{2} - \lambda_s\right),\ \cos(\pi - \beta) = -\cos\beta,\ \cos\left(\frac{\pi}{2} - \lambda_p\right) = \sin\lambda_p,$$

$$\text{and } \sin\left(\frac{\pi}{2} - \lambda_p\right) = \cos\lambda_p \tag{A.37}$$

into Equations (A.35) and (A.36) to yield

$$\cos p = \sin\lambda_p \sin\lambda_p + \cos\lambda_p \cos\lambda_p \cos\beta \tag{A.38}$$

for the smaller triangle and

$$\cos p = \sin\lambda_p \sin\lambda_s - \cos\lambda_p \cos\lambda_s \cos\beta \tag{A.39}$$

for the larger triangle.

At this point we realize that λ_p, λ_s, and β are all limited to the range $-\pi/2$ to $+\pi/2$. Within this range, the product $\cos\lambda_p \cos\lambda_s \cos\beta$ will always be positive. So if we find $\cos p \geq \sin\lambda_p \sin\lambda_s$, this indicates that we must be dealing with the smaller spherical triangle in Figure A.3b, and pole longitude is given by

$$\phi_p = \phi_s + \beta \tag{A.40}$$

But if we find $\cos p < \sin\lambda_p \sin\lambda_s$, we must be dealing with the larger triangle in Figure A.3b, for which

$$\phi_p = \phi_s + \pi - \beta \tag{A.41}$$

This development explains the conditional tests and alternative methods of calculating ϕ_p given by Equations (7.4) through (7.7).

CONFIDENCE LIMITS ON POLES: *dp* AND *dm*

From the previous section, we know how to map an observed magnetic field direction I and D observed at site(λ_s, ϕ_s) into a magnetic pole position (λ_p, ϕ_p). Now we consider the confidence limits on (λ_p, ϕ_p) resulting from circular confidence limits on the direction.

We start by determining the confidence limits, ΔI, on the inclination and on the declination, ΔD, from I, D, and α_{95} (the usual confidence limit on the direction). At this point, this is a direction space problem as schematically represented on the lower hemisphere of the equal-area projection in Figure A.4a. Two examples of directions and confidence limits are shown in this diagram. Note how a steep inclination results in a large confidence limit ΔD on the declination.

Now consider the spherical triangle ABC of Figure A.4a. The angular distance $b = (\pi/2) - I$ and $c = \alpha_{95}$. The angle B is $\pi/2$, and the angle C is ΔD. Apply the Law of Sines to this triangle to give

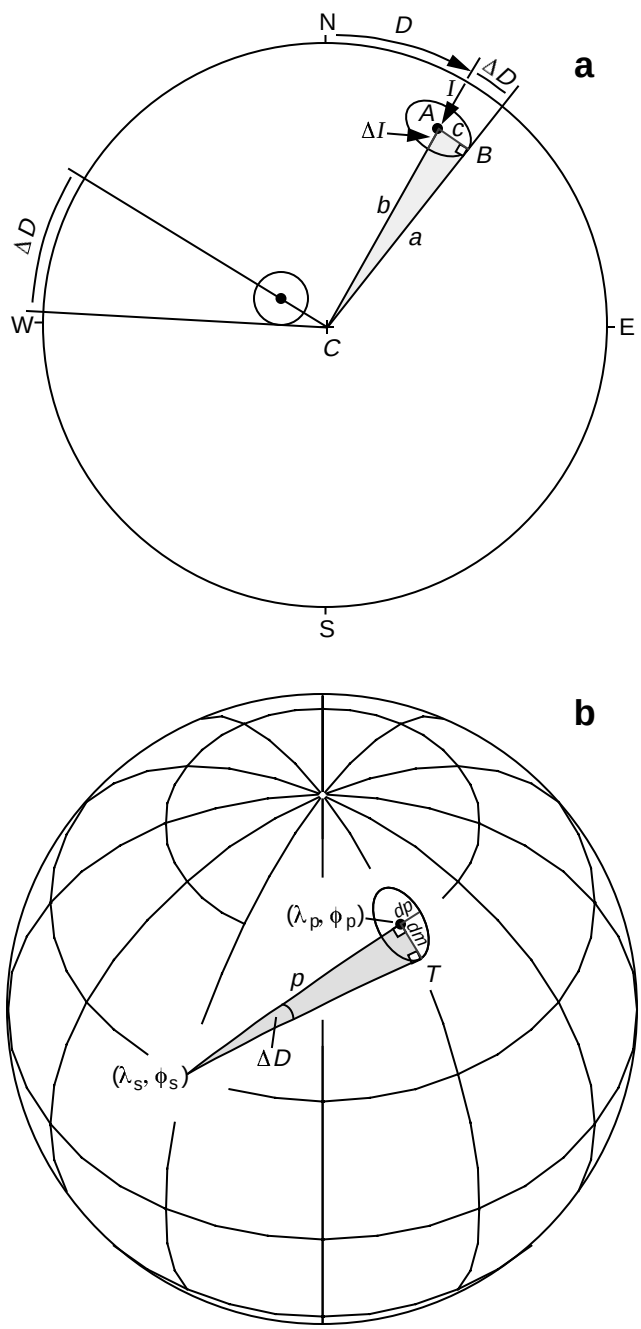

Figure A.4 (**a**) Equal-area projection of direction I, D and attendant confidence limits ΔI, ΔD. The confidence limit surrounding the direction is circular in direction space but is mapped into an ellipse by the equal-area projection. (**b**) Magnetic pole at (λ_p, ϕ_p) and attendant confidence limits dp and dm. The site location is (λ_s, ϕ_s); p is the magnetic colatitude; the dark stippled region is a spherical triangle with apices (λ_s, ϕ_s), (λ_p, ϕ_p), and T; the light stippled region is a confidence oval about the magnetic pole with semi-major and semi-minor axes dm and dp, respectively; ΔD is the angle at apex (λ_s, ϕ_s).

$$\frac{\sin c}{\sin C} = \frac{\sin b}{\sin B} \tag{A.42}$$

which rearranges to

$$\sin C = \frac{\sin c \sin B}{\sin b} \tag{A.43}$$

Substituting the above quantities for b, c, B, and C yields

$$\sin \Delta D = \frac{\sin \alpha_{95} \sin \frac{\pi}{2}}{\sin\left(\frac{\pi}{2} - I\right)} = \frac{\sin \alpha_{95}}{\cos I} \tag{A.44}$$

from which ΔD can be determined. By inspection of Figure A.4a,

$$\Delta I = \alpha_{95} \tag{A.45}$$

Now we turn our attention to Figure A.4b, which illustrates mapping a magnetic field direction I, D observed at (λ_s, ϕ_s) into a magnetic pole at (λ_p, ϕ_p). Consider the spherical triangle with apices at (λ_s, ϕ_s), (λ_p, ϕ_p), and T. The angle at apex (λ_s, ϕ_s) is ΔD. The angles at apices (λ_p, ϕ_p) and T are both $\pi / 2$. The angular distance from (λ_s, ϕ_s) to (λ_p, ϕ_p) is the magnetic colatitude p. The angular distance from (λ_p, ϕ_p) to T is dm, the confidence limit perpendicular to the great-circle path from (λ_s, ϕ_s) to (λ_p, ϕ_p.).

Apply the Law of Sines to get

$$\frac{\sin dm}{\sin \Delta D} = \frac{\sin p}{\sin T} \tag{A.46}$$

Now substitute

$$T = \frac{\pi}{2} \text{ and } \sin \Delta D = \frac{\sin \alpha_{95}}{\cos I}$$

(from Equation (A.44)) and rearrange to get

$$dm = \sin^{-1}\left(\frac{\sin \alpha_{95} \sin p}{\cos I}\right) \tag{A.47}$$

This is the general expression for the confidence limit dm. But because dm and α_{95} are usually small angles and sin (x) ≈ x, for small x, Equation (A.47) is usually given as

$$dm = \alpha_{95} \frac{\sin p}{\cos I} \tag{A.48}$$

which is Equation (7.9).

From Equation (A.10), we know that

$$p = \tan^{-1}\left(\frac{2}{\tan I}\right) = \cot^{-1}\left(\frac{\tan I}{2}\right) \tag{A.49}$$

Now we use

$$d\left(\cot^{-1} x\right) = -\frac{dx}{1+x^2} \quad \text{and} \quad d(\tan x) = \sec^2 x \, dx$$

to get

$$dp = d\left[\cot^{-1}\left(\tfrac{1}{2}\tan I\right)\right] = -\frac{\frac{1}{2}\sec^2 I \, dI}{1+\frac{1}{4}\tan^2 I} = -\frac{2\sec^2 I \, dI}{4+\tan^2 I} \tag{A.50}$$

Use of the trigonometric identities

$$\sec^2 x = \frac{1}{\cos^2 x}, \ \tan x = \frac{\sin x}{\cos x}, \text{ and } \sin^2 x + \cos^2 x = 1$$

in Equation (A.50) yields

$$dp = \frac{-2\dfrac{dI}{\cos^2 I}}{4+\dfrac{\sin^2 I}{\cos^2 I}} = \frac{-2dI}{4\cos^2 I + \sin^2 I} = \frac{-2dI}{1+3\cos^2 I} \tag{A.51}$$

Recalling that $dI = \alpha_{95}$ and observing that dp is symmetrical about (λ_p, ϕ_p) give the end result

$$dp = 2\alpha_{95}\left(\frac{1}{1+3\cos^2 I}\right) \tag{A.52}$$

which is Equation (7.8).

EXPECTED MAGNETIC FIELD DIRECTION

The problem here is to derive expressions that allow determination of the magnetic field direction expected at an observing site (λ_s, ϕ_s) due to a geocentric dipole with pole position (λ_p, ϕ_p). We also derive expressions for the confidence limits on the expected direction that result from circular confidence limits (usually A_{95}) on the pole. The geometry of the problem is illustrated in Figure A.5.

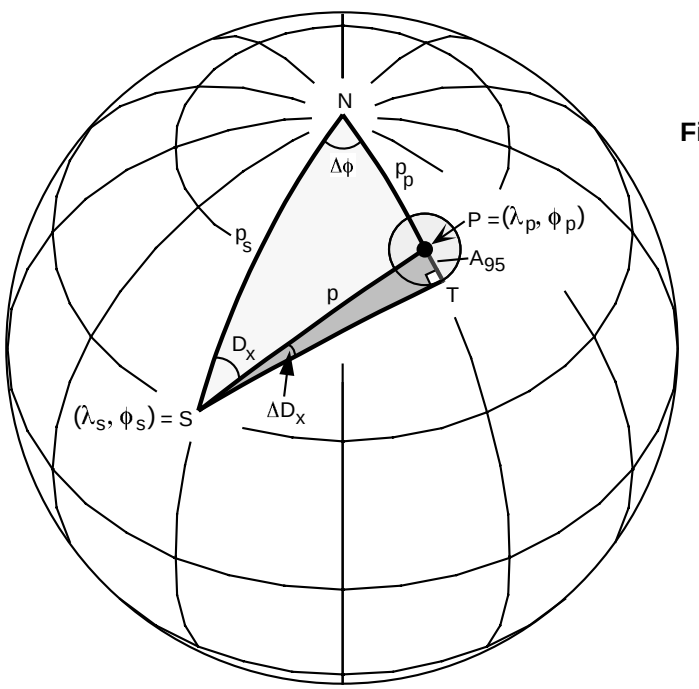

Figure A.5 Geometry used to determine the expected magnetic field direction from a magnetic pole. The magnetic pole is at P (λ_p, ϕ_p) with circular confidence limit A_{95}; the site location is at S (λ_s, ϕ_s); N is the geographic north pole; p is the magnetic colatitude of the site; p_s is the geographic colatitude of the site; p_p is the geographic colatitude of the magnetic pole; $\Delta\phi$ is the longitudinal difference between the magnetic pole and the site; D_x is the expected magnetic field declination at the site with confidence limit = ΔD_x.

A spherical triangle SPN is constructed with N at the geographic pole, P at the magnetic pole (λ_p, ϕ_p), and S at the site (λ_s, ϕ_s). Having gone through a similar problem before, we realize that the declination of the expected magnetic field direction, D_x, at site (λ_s, ϕ_s) is the angle at apex S.

The first step in the derivation is to determine the angular distance, p, from (λ_p, ϕ_p) to (λ_s, ϕ_s). Apply the law of sines to triangle SPN in Figure A.5 to get

$$\cos p = \cos p_p \cos p_s + \sin p_p \sin p_s \cos \Delta\phi \tag{A.53}$$

Now substitute

$$p_p = \frac{p}{2} - \lambda_p \ , \quad p_s = \frac{\pi}{2} - \lambda_s \ , \ \text{and} \ \Delta\phi = \phi_p - \phi_s$$

into Equation (A.53) and use

$$\cos\left(\frac{\pi}{2} - \lambda_p\right) = \sin\lambda_p \ \text{and} \ \sin\left(\frac{\pi}{2} - \lambda_p\right) = \cos\lambda_p$$

to find

$$\cos p = \sin\lambda_p \sin\lambda_s + \cos\lambda_p \cos\lambda_s \cos\left(\phi_p - \phi_s\right) \tag{A.54}$$

from which you can determine p.

The expected inclination, I_x, can be determined from p by using the dipole equation (Equation (A.10)):

$$I_x = \tan^{-1}\left(2\cot p\right) \tag{A.55}$$

The confidence limit on I_x is defined as ΔI_x and can be determined from the equation that we derived to get dp from ΔI (= α_{95} in Equation (A.52)) <u>and</u> substituting A_{95} for dp:

$$A_{95} = 2\Delta I_x \left(\frac{1}{1+3\cos^2 I_x}\right) \tag{A.56}$$

which rearranges to give

$$\Delta I_x = \frac{A_{95}}{2}\left(1+3\cos^2 I_x\right) = A_{95}\left(\frac{2}{1+3\cos^2 p}\right) \tag{A.57}$$

To determine the expected declination, D_x, we can use Equation (A.28) derived above:

$$\sin\lambda_p = \sin\lambda_s \cos p + \cos\lambda_s \sin p \cos D \tag{A.28}$$

and rearrange to solve for D_x:

$$\cos D_x = \frac{\sin\lambda_p - \sin\lambda_s \cos p}{\cos\lambda_s \sin p} \tag{A.58}$$

from which D_x can be determined.

The confidence limit on D_x is ΔD_x, which can be derived by applying the Law of Sines to the spherical triangle STP (Figure A.5):

$$\frac{\sin A_{95}}{\sin \Delta D_x} = \frac{\sin p}{\sin T} \tag{A.59}$$

Now note that $T = \pi / 2$ (thus $\sin T = 1$) and rearrange to give

$$\Delta D_x = \sin^{-1}\left(\frac{\sin A_{95}}{\sin p}\right) \tag{A.60}$$

If you actually go through some calculations of ΔI_x and ΔD_x, you will find that these confidence limits change with I_x (and p) in a systematic fashion. For small p (steep inclination), $\Delta D_x > \Delta I_x$; $\Delta I_x \approx \Delta D_x$ at about $p = 60°$ ($I_x \approx 50°$); for $60° < p \leq 90°$ ($I_x < 50°$), $\Delta I_x > \Delta D_x$.

This determination of the confidence limits ΔI_x and ΔD_x produces a confidence *oval* (not circle) about I_x, D_x. ΔI_x is the semi-axis of the confidence oval in the vertical plane through I_x, D_x. But the other semi-axis of the confidence oval is not ΔD_x. Remember that ΔD_x is the projection of the direction space confidence limit onto the periphery of the equal-area projection (Figure A.4a). The required dimension of the confidence limit about I_x, D_x can be determined from Equation (A.44) by substituting the desired angular distance c (Figure A.4a) for α_{95}. This leads to

$$c = \sin^{-1}\left(\cos I_x \sin \Delta D_x\right) \tag{A.61}$$

ROTATION AND FLATTENING IN DIRECTION SPACE

Here, we derive the equations to evaluate the vertical axis rotation that is required to align an observed declination with an expected declination. In addition, we develop equations to determine the *flattening of inclination* indicated by comparison of the observed and expected inclination.

The equal-area projection of Figure A.6 illustrates the problem. In this example, the observed direction has inclination $I_o = 40°$, and declination $D_o = 20°$. The confidence limit is $\alpha_{95} = 8°$. This observed direction is compared to an expected direction at the sampling site, $I_x = 60°$ and $D_x = 330°$. The confidence limits on the expected direction are $\Delta I_x = 5.3°$ and $\Delta D_x = 8°$.

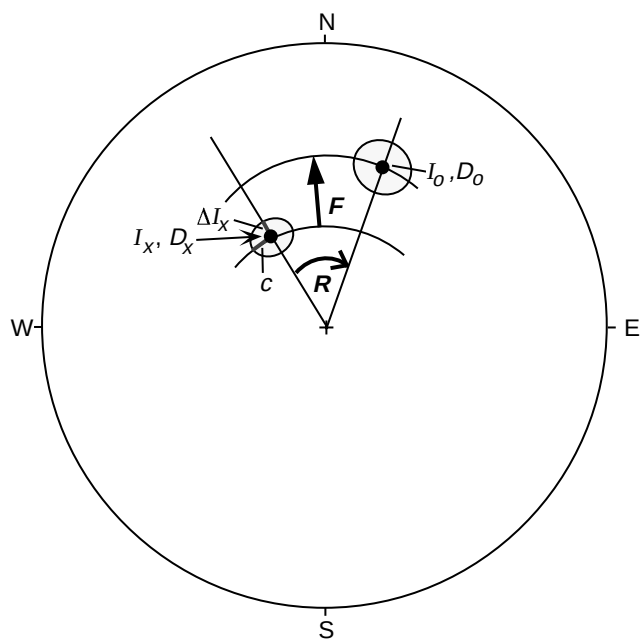

Figure A.6 Equal-area projection of vertical axis rotation, *R*, and inclination flattening, *F*. The observed direction is I_o, D_o; the expected direction is I_x, D_x; the confidence regions about the directions are shown by stippling; *c* is the angular distance of the confidence limit from the vertical plane through I_x, D_x.

The vertical-axis rotation is *R* and is defined as positive for an observed direction clockwise from the expected direction as shown in Figure A.6. The vertical-axis rotation is simply given by

$$R = D_o - D_x \tag{A.62}$$

The flattening of inclination is labeled *F* and is defined as positive when the observed inclination is less than ("flatter" than) the expected inclination. Thus *F* is given simply by

$$F = I_x - I_o \tag{A.63}$$

We need a method to evaluate confidence limits on *R* and *F*, which are labeled ΔR and ΔF, respectively. The original method of assigning confidence limits to *R* and *F* was to treat the errors in the observed and expected directions as independent errors. This approach led to

$$\Delta R = \sqrt{\Delta D_o^2 + \Delta D_x^2} \tag{A.64}$$

and

$$\Delta F = \sqrt{\Delta I_o^2 + \Delta I_x^2} \tag{A.65}$$

The confidence limits ΔD_o, and ΔI_o can be determined from Equations (A.44) and (A.45). ΔI_x and ΔD_x can be determined from Equations (A.57) and (A.60). Subsequent to derivation of the above equations, a rigorous statistical analysis of the confidence limits on *R* and *F* by Demarest (1983; reference in Chapter 11) has shown that the confidence limits should be calculated by using the following equations:

$$\Delta R = 0.8\sqrt{\Delta D_o^2 + \Delta D_x^2} \tag{A.66}$$

and

$$\Delta F = 0.8\sqrt{\Delta I_o^2 + \Delta I_x^2} \tag{A.67}$$

ROTATION AND POLEWARD TRANSPORT IN POLE SPACE

Rotation about a vertical axis and (paleo)latitudinal transport are sometimes more effectively addressed by comparing an observed paleomagnetic pole with a reference paleomagnetic pole. This situation is shown in Figure A.7. The reference pole is at point *RP* (λ_r, ϕ_r) with $A_{95} = A_r$; the observed pole is at point *OP* (λ_o, ϕ_o) with $A_{95} = A_o$; the site location from which the observed pole was determined is at point *S* (λ_s, ϕ_s). The problem is to determine the vertical axis rotation, *R*, and the poleward transport (motion toward the reference pole) indicated by the discordance between the observed pole and the reference pole.

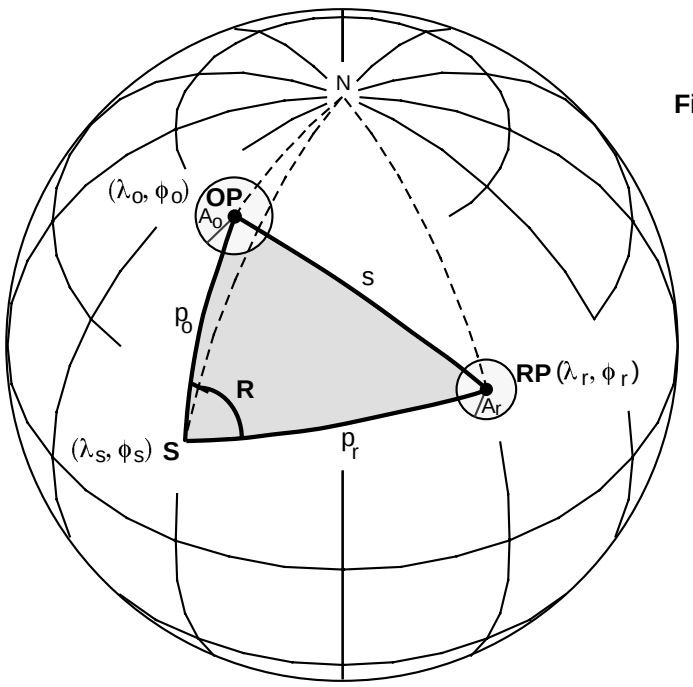

Figure A.7 Geometry required to determine vertical axis rotation and poleward displacement by comparing observed and reference paleomagnetic poles. *RP* is the reference paleomagnetic pole at (λ_r, ϕ_r) with $A_{95} = A_r$; *OP* is the observed paleomagnetic pole at (λ_o, ϕ_o) with $A_{95} = A_o$; the site location *S* is (λ_s, ϕ_s); *N* is the north geographic pole; the dashed longitudinal lines connect *S*, *OP*, and *RP* to *N*; the dark stippled region is a spherical triangle with apices *S*, *OP*, *RP* and sides p_o, p_r, and *s*; the light stippled circles are confidence circles about observed and reference poles; the vertical axis rotation is angle *R*.

We form the spherical triangle shown in Figure A.7 with apices at *S*, *OP*, and *RP*. The first step is to determine the angular distances p_r, p_o, and *s*. There are two approaches: (1) Use the formula developed for determining the great-circle distance between two locations (Equation (A.20)); or (2) use the formula developed for determining the angular distance from observation location to the observed paleomagnetic pole (Equation (A.38)). For the second approach, we form three spherical triangles (*N–OP–S*, *N–S–RP*, and *N–OP–RP*) by connecting the three apices of *S–OP–RP* with the geographic pole. Equation (A.38) is then applied to each of these three triangles to determine the unknown angular distances p_r, p_o, and *s*. The results are

$$p_r = \cos^{-1}\left(\sin\lambda_r \sin\lambda_s + \cos\lambda_r \cos\lambda_s \cos\left[\phi_r - \phi_s\right]\right) \tag{A.68}$$

$$p_o = \cos^{-1}\left(\sin\lambda_s \sin\lambda_o + \cos\lambda_s \cos\lambda_o \cos\left[\phi_s - \phi_o\right]\right) \tag{A.69}$$

$$s = \cos^{-1}\left(\sin\lambda_r \sin\lambda_o + \cos\lambda_r \cos\lambda_o \cos\left[\phi_r - \phi_o\right]\right) \tag{A.70}$$

Knowing these angular distances, we can determine the rotation, *R*, by realizing that *R* is the angle at apex *S* and applying the Law of Cosines to the spherical triangle *S–OP–RP*:

$$\cos s = \cos p_o \cos p_r + \sin p_o \sin p_r \cos R \tag{A.71}$$

Solving for *R* gives

$$R = \cos^{-1}\left(\frac{\cos s - \cos p_o \cos p_r}{\sin p_o \sin p_r}\right) \tag{A.72}$$

Note that Equation (A.72) will not tell you whether *R* is positive (clockwise rotation) or negative (counter-clockwise rotation). But inspection of Figure A.7 indicates that *R* is negative in this example.

The poleward transport, *p*, is simply

$$p = p_o - p_r \tag{A.73}$$

The confidence limit on *R* can be determined from Equation (A.66):

$$\Delta R = 0.8\sqrt{\Delta D_o^2 + \Delta D_x^2} \tag{A.66}$$

where from Equation (A.60):

$$\Delta D_x = \sin^{-1}\left(\frac{\sin A_r}{\sin p_r}\right) \tag{A.74}$$

and

$$\Delta D_o = \sin^{-1}\left(\frac{\sin A_o}{\sin p_o}\right) \tag{A.75}$$

The confidence limit on *p* is Δp and is given by

$$\Delta p = 0.8\sqrt{\Delta p_r^2 + \Delta p_o^2} \tag{A.76}$$

From inspection of Figure A.7, we can see that

$$\Delta p_o = A_o \tag{A.77}$$

and

$$\Delta p_r = A_r \tag{A.78}$$

PALEOLATITUDES AND CONFIDENCE LIMITS

A paleogeographic map is often used to compare the paleomagnetically determined paleolatitude of an accreted terrane with the paleolatitude of the continent to which the terrane was accreted. The confidence limits on the paleolatitude of the terrane are illustrated by showing the upper and lower paleolatitudinal limits. An example is shown in Figure 11.13. In this section, we derive the equations that are used to determine paleolatitudes and the attendant confidence limits.

Two basic approaches to this problem have been used in the paleomagnetic literature. As with the rotation and transport problem, one approach uses the observed paleomagnetic direction, while the other uses the observed paleomagnetic pole. We'll first derive the equations for the direction-space approach then address the pole-space approach.

If we observe a mean paleomagnetic inclination I_o at a particular site, the dipole equation (Equation A.11) can be used to determine the paleolatitude:

$$\lambda_o = \tan^{-1}\left(\frac{\tan I_o}{2}\right) \tag{A.79}$$

The confidence limit on I_o is $\Delta I_o = \alpha_{95}$. Because of the nonlinearity of Equation (A.79), the resulting confidence limits on λ_o are not symmetric about λ_o. Adding $\Delta I_o = \alpha_{95}$ to I_o will yield the higher latitude confidence limit, which we can label λ_o^+:

$$\lambda_o^+ = \tan^{-1}\left[\frac{\tan\left(I_o + \alpha_{95}\right)}{2}\right] \tag{A.80}$$

The lower confidence limit λ_o^- is determined by subtracting $\Delta I_o = \alpha_{95}$ from I_o:

$$\lambda_o^- = \tan^{-1}\left[\frac{\tan\left(I_o - \alpha_{95}\right)}{2}\right] \tag{A.81}$$

These confidence limits on paleolatitude λ_o will be symmetric about λ_o only for $\lambda_o = 0°$ or $\lambda_o = 90°$.

This derivation explains why paleolatitudes determined from paleomagnetic inclinations are sometimes listed with asymmetric confidence limits. For example, "The Cretaceous paleolatitude of the Macintosh Terrane is 42.3° with upper and lower 95% confidence limits of 50.0° and 35.7°, respectively."

In the pole-space approach, the paleogeographic map for a continent is produced as described in Chapter 10. The confidence limit for the reference pole is A_r, which directly gives the paleolatitude confidence limit for any point on the continent. As explained in the development of rotation and poleward displacement of a crustal block (and illustrated in Figure A.7), the angular distance from the site location to the observed pole is p_o, the observed paleocolatitude. From p_o, the observed paleolatitude is easily determined by

$$\lambda_o = 90° - p_o \tag{A.82}$$

The confidence limit on λ_o is simply A_o, the confidence limit on the observed pole (= confidence limit on p_o; Equation (A.77)). So there are confidence limits on the paleolatitude of the crustal block and on the continent to which the terrane is now attached.

The simplest way to make a paleogeographic map that encompasses these paleolatitudinal confidence limits is to use the results derived for poleward transport. You want to show how far the crustal block has moved latitudinally with respect to the continent. So you place the continent in its paleogeographic position; then use Equation (A.82) to determine the paleolatitude of the crustal block and place the block at that paleolatitude. The confidence limit on paleolatitudinal position of the crustal block <u>with respect to the continent</u> is the confidence limit Δp on poleward transport (Equation (A.76)). This confidence limit accounts for uncertainties in the paleolatitudes of both the crustal block and the continent. But to make the paleogeographic map, we fix the continent in the paleogeographic grid and ascribe all the paleolatitudinal uncertainty to the crustal block. The confidence limits on paleolatitude of the block are shown as $\lambda_o \pm \Delta p$ on the paleogeographic map. This procedure was used to construct the Middle–Late Triassic paleogeographic map of Figure 11.13 showing the paleolatitude of the Nikolai Greenstone.