Lag-Lead Compensator Design Using Bode Plots

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These notes are lecture notes prepared by Prof. Guy Beale for presentation in ECE 421, *Classical Systems and Control Theory*, in the Electrical and Computer Engineering Department, George Mason University, Fairfax, VA. Additional notes can be found at: http://teal.gmu.edu/~gbeale/examples.html.

I. INTRODUCTION

As with phase lag and phase lead compensation, the purpose of lag-lead compensator design in the frequency domain generally is to satisfy specifications on steady-state accuracy and phase margin. Typically, there is also a specification (implicitly or explicitly) on gain crossover frequency or closed-loop bandwidth. A phase margin specification can represent a requirement on relative stability due to pure time delay in the system, or it can represent desired transient response characteristics that have been translated from the time domain into the frequency domain. A specification on bandwidth or crossover frequency can represent a requirement on speed of response in the time domain or a frequency-domain requirement on which sinusoidal frequencies will be passed by the system without significant attenuation.

The overall philosophy in the design procedure presented here is for the lead part of the compensator to adjust the system's Bode phase curve to establish the required phase margin at a specified frequency, without reducing the zero-frequency magnitude value. The lag part of the compensator is used to drop the magnitude curve down to 0 db at that specified frequency. The lag compensator must attenuate the magnitude of the series combination of the lead compensator $G_{c\_lead}(s)$ and the plant $G_p(s)$ at the chosen frequency. Thus, in the procedure presented here, the lead compensator is designed first. In order for lag-lead compensation to work in this context, the following two characteristics are needed:

- the uncompensated phase shift at the chosen gain crossover frequency must be more negative than the value needed to satisfy the phase margin specification (otherwise, no lead compensation is needed);
- the Bode magnitude curve (after the lead compensator has been designed) must be above 0 db at the frequency chosen for the gain crossover frequency (otherwise no lag compensation is needed, just additional gain).

The basic lag-lead compensator has two stages, one each of lag and lead compensation. If the compensator is to have a single-stage lead compensator, then the amount that the phase curve needs to be moved up at the gain crossover frequency in order to satisfy the phase margin specification must be less than $90°$, and is generally restricted to a maximum value in the range $55°$–$65°$. Multiple stages of lead compensation can be used, following the same procedure as shown below, and are needed when the amount that the Bode phase curve must be moved up exceeds the available phase shift for a single stage of compensation.

If the compensator is to have a single-stage lag compensator, then it must be possible to drop the magnitude curve down to 0 db at the gain crossover frequency without using excessively large component values. Multiple stages of compensation can be used, following the same procedure as shown below. Multiple stages are needed when the amount that the Bode magnitude curve must be moved down is too large for a single stage of compensation. More is said about this later.

The gain crossover frequency and bandwidth for the lag-lead-compensated system may be higher or lower than for the plant itself or for the plant after the steady-state error specification is satisfied. This depends on the value chosen for the compensated gain crossover frequency. The higher the crossover frequency, the more rapidly the system will respond in the time domain. The faster response may be an advantage in many applications, but a disadvantage of a wider bandwidth is that more noise and other high frequency signals (often unwanted) will be passed by the system. A smaller bandwidth also provides more stability robustness when the system has unmodeled high frequency dynamics, such as the bending modes in aircraft and spacecraft. Thus, there is a trade-off between having the ability to track rapidly varying reference signals and being able to reject high-frequency disturbances.

The design procedure presented here is basically graphical in nature. All of the measurements needed can be obtained from accurate Bode plots of the uncompensated system. If data arrays representing the magnitudes and phases of the system at various frequencies are available, then the procedure can be done numerically, and in many cases automated. The examples and plots presented in theses notes are all done in MATLAB, and the various measurements that are presented in the examples are obtained from the relevant data arrays.

The primary references for the procedures described in theses notes are [1]–[3]. Other references that contain similar material include [4]–[11].

Since detailed discussions on the design of lead and lag compensators individually have been presented in my notes[1] "*Phase Lag Compensator Design Using Bode Plots*" and "*Phase Lead Compensator Design Using Bode Plots*", theses notes will concentrate on the use of those two designs together. The reader is referred to those notes for details on the individual design procedures. The procedure for designing the lag-lead compensator is presented in Section II. A simple example is used throughout that section to illustrate the procedure. A more advanced example is presented in Section III.

II. DESIGN PROCEDURE

A. Compensator Structure

The basic lag-lead compensator consists of a gain, two poles, and two zeros, and from a transfer function standpoint is just the series combination of a lag compensator and a lead compensator. Based on the usual electronic implementation of those compensators [3], the specific structure of the lag-lead compensator is:

[1] See http://teal.gmu.edu/~gbeale/ece_421/examples_421.html

$$G_{c\_lag\_lead}(s) = K_c \cdot \left[\frac{1}{\alpha_d} \cdot \frac{(s+z_{cd})}{(s+p_{cd})}\right] \cdot \left[\frac{1}{\alpha_g} \cdot \frac{(s+z_{cg})}{(s+p_{cg})}\right] \tag{1}$$
$$= K_c \frac{(s/z_{cd}+1)}{(s/p_{cd}+1)} \cdot \frac{(s/z_{cg}+1)}{(s/p_{cg}+1)}$$
$$= K_c \frac{(\tau_d s+1)}{(\alpha_d \tau_d s+1)} \cdot \frac{(\tau_g s+1)}{(\alpha_g \tau_g s+1)}$$

with

$$\begin{aligned} z_{cd} &> 0, \quad p_{cd} > 0, \quad \alpha_d \triangleq \frac{z_{cd}}{p_{cd}} < 1, \quad \tau_d = \frac{1}{z_{cd}} = \frac{1}{\alpha_d p_{cd}} \\ z_{cg} &> 0, \quad p_{cg} > 0, \quad \alpha_g \triangleq \frac{z_{cg}}{p_{cg}} > 1, \quad \tau_g = \frac{1}{z_{cg}} = \frac{1}{\alpha_g p_{cg}} \end{aligned} \tag{2}$$

The subscript d on the various variables indicates the lead compensator, and the subscript g indicates the lag compensator.

Figure 1 shows the Bode plots of magnitude and phase for a typical lag-lead compensator. The values in this example are $K_c = 1$, $p_{cg} = 0.032$, $z_{cg} = 0.2$, $z_{cd} = 0.8$, and $p_{cd} = 5$, so $\alpha_g = 0.2/0.032 = 6.25$ and $\alpha_d = 0.8/5 = 0.16$. Changing the gain merely moves the magnitude curve by $20 * \log_{10}|K_c|$. The choice of $\alpha_d = 1/\alpha_g$ for these plots was made for convenience; it is not a requirement for the design procedure. Some textbooks present design procedures that require $\alpha_d = 1/\alpha_g$, but that will not be done in theses notes.

The major characteristics of the lag-lead compensator are the magnitude attenuation in the intermediate frequencies and the positive phase shift at slightly higher frequencies. The maximum positive phase shift occurs at the frequency $\omega = \omega_{\max}$, which is the geometric mean of z_{cd} and p_{cd}. The minimum value in the magnitude curve occurs (at least approximately) at the frequency that is the geometric mean of z_{cg} and z_{cd}. The large negative phase shift that is seen at intermediate frequencies is undesired but unavoidable. Proper design of the compensator requires placing the compensator poles and zeros appropriately so that the benefits of the positive phase shift and the magnitude attenuation are obtained at the correct frequency, without the negative phase shift causing problems. The following paragraphs show how this can be accomplished.

B. Outline of the Procedure

The procedure for designing the lag-lead compensator is almost the same as designing a lead compensator and then designing a lag compensator. There are only small differences in the steps required for the lead and lag designs. These differences allow us to take into account the interaction between the lag and lead parts of the compensator.

The following steps outline the procedure that will be used to design the lag-lead compensator to satisfy steady-state error, phase margin, and gain crossover frequency specifications. If the specification is given in terms of the closed-loop bandwidth ω_B instead of the gain crossover frequency ω_x, the following rule of thumb can be used to produce a preliminary design: $\omega_B \approx 1.5\omega_x$.

1) Determine if the System Type N needs to be increased in order to satisfy the steady-state error specification, and if necessary, augment the plant with the required number of poles at $s = 0$. Calculate K_c to satisfy the steady-state error.
2) Make the Bode plots of $G(s) = K_c G_p(s)/s^{(N_{req} - N_{sys})}$.
3) Design the lead portion of the lag-lead compensator:
   a) determine the amount of phase shift in $G(j\omega)$ at the specified gain crossover frequency and calculate the uncompensated phase margin $PM_{uncompensated}$ (assuming that the specified gain crossover frequency defines the uncompensated phase margin);
   b) calculate the values of $\phi_{\max}$ and α_d that are required to raise the phase curve to the value needed to satisfy the phase margin specification;
   c) using the value of α_d and the specified gain crossover frequency, compute the lead compensator's zero z_{cd} and pole p_{cd}.
4) Design the lag portion of the lag-lead compensator:
   a) determine the magnitude of $G(j\omega)$ at the specified gain crossover frequency;
   b) determine the amount of shift to the magnitude curve at the specified gain crossover frequency that is caused by the lead compensator;
   c) determine the amount of magnitude attenuation that is required to drop the combined (plant + lead compensator) magnitude down to 0 db, and compute the corresponding α_g.
   d) using the value of α_g and the specified gain crossover frequency, compute the lag compensator's zero z_{cg} and pole p_{cg}.
5) If necessary, choose appropriate resistor and capacitor values to implement the compensator design.

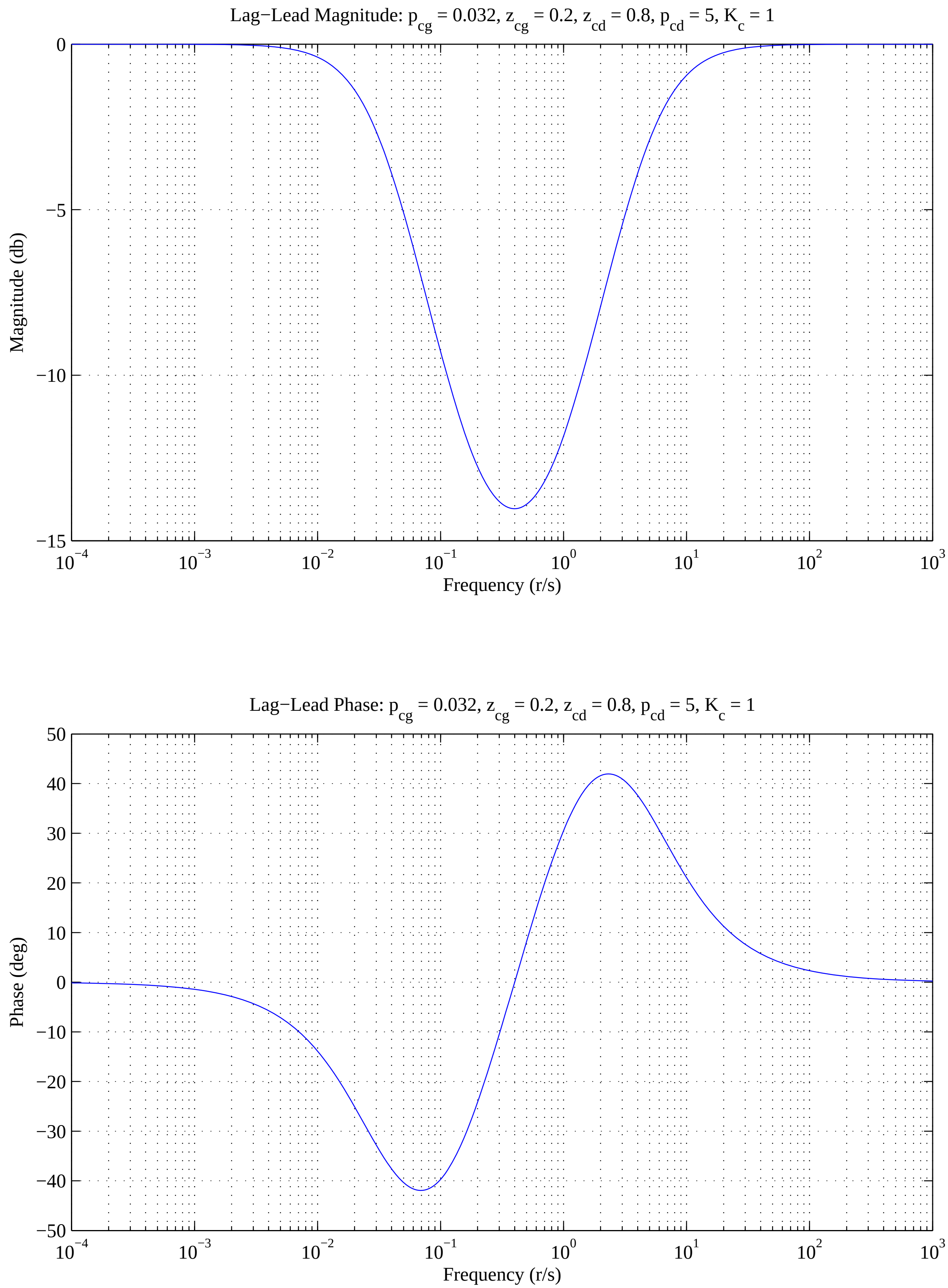

Fig. 1. Bode magnitude and phase plots for a typical lag-lead compensator.

To illustrate the design procedure, the following system model and specifications will be used:

$$G_p(s) = \frac{2}{s} \tag{3}$$

- steady-state error specification for a unit *parabolic* input is $e_{ss\_specified} = 0.0125$;
- phase margin specification is $PM_{specified} \geq 45^\circ$;
- gain crossover frequency specification $\omega_{x\_compensated} \approx 5$ r/s.

C. Compensator Gain

The first step in the design procedure is to determine the number of poles that the compensator must have at $s = 0$ and the value of the gain K_c. Since the steady-state error specification is for a parabolic input, the required number of poles at the origin in the compensated system is $N_{req} = 2$. The plant model described in (3) has $N_{sys} = 1$, so the compensator must have $N_{req} - N_{sys} = 2 - 1 = 1$ pole located at $s = 0$.

When $G_p(s)$ is augmented with this compensator pole at the origin, the error constant of $G_p(s)/s$ is $K_{x\_plant} = 2/1 = 2$, so the steady-state error for a parabolic input is $e_{ss\_plant} = 1/2 = 0.5$. The specified steady-state error $e_{ss\_specified} = 0.0125$ requires an error constant of $K_{x\_required} = 1/0.0125 = 80$. Therefore, the compensator requires a gain having a value of

$$\begin{aligned} K_c &= \frac{e_{ss\_plant}}{e_{ss\_specified}} = \frac{0.5}{0.0125} = 40 \\ &= \frac{K_{x\_required}}{K_{x\_plant}} = \frac{80}{2} = 40 \end{aligned} \tag{4}$$

so that

$$\frac{K_c}{s^{(N_{req} - N_{sys})}} = \frac{40}{s} \tag{5}$$

D. Making the Bode Plots

The next step is to plot the magnitude and phase as a function of frequency ω for the series combination of the compensator gain (and any compensator poles at $s = 0$) and the given system $G_p(s)$. This transfer function will be the one used to determine the values of the compensator's poles and zeros and to determine if more than one stage of either lead or lag compensation is needed. The magnitude $|G(j\omega)|$ is generally plotted in decibels (db) vs. frequency on a log scale, and the phase $\angle G(j\omega)$ is plotted in degrees vs. frequency on a log scale. At this stage of the design, the system whose frequency response is being plotted is

$$\begin{aligned} G(s) &= \frac{K_c}{s^{(N_{req} - N_{sys})}} \cdot G_p(s) \\ &= \frac{40}{s} \cdot \frac{2}{s} = \frac{80}{s^2} \end{aligned} \tag{6}$$

Figure 2 shows the Bode plots for this augmented system. The dashed lines are the curves for $G_p(j\omega)$. Since the compensator does have one pole at the origin, the slopes of the magnitude curves for $G(j\omega)$ and $G_p(j\omega)$ differ by -20 db/decade at all frequencies. At $\omega = 1$ r/s, the gain K_c has raised the curve by $20 \log_{10} |40| \approx 32$ db. The plant's phase curve is shifted by -90° at all frequencies in producing $\angle G(j\omega)$, due to the compensator's pole at the origin.

The next phase of the design deals with determining $(s/z_{cd} + 1)/(s/p_{cd} + 1)$. The values of z_{cd} and p_{cd} will be chosen to satisfy the phase margin specification at the required gain crossover frequency. Note that at $\omega = 0$, the magnitude $|(j\omega/z_{cd} + 1)/(j\omega/p_{cd} + 1)| = 1 \Rightarrow 0$ db and the phase $\angle (j\omega/z_{cd} + 1)/(j\omega/p_{cd} + 1) = 0$ degrees. Therefore, the low-frequency parts of the curves just plotted will be unchanged, and the steady-state error specification will remain satisfied. The Bode plots after the design of the lead part of the compensator will be the sum, at each frequency, of the plots made in this step of the procedure and the plots of $(j\omega/z_{cd} + 1)/(j\omega/p_{cd} + 1)$.

E. Uncompensated Phase Margin

Since the purpose of the lead compensator is to move the phase curve upwards in order to satisfy the phase margin specification, we need to determine how much positive phase shift is required. The first step in this determination is to evaluate the phase margin of the given system in (6). The uncompensated phase margin is

$$PM_{uncompensated} = 180^\circ + \angle G(j\omega_x) \tag{7}$$

Since the system in (6) is Type 2, and there are no other poles or zeros, the phase shift of $G(j\omega)$ is -180° at all frequencies. Therefore, the uncompensated phase margin is $PM_{uncompensated} = 180^\circ + (-180^\circ) = 0^\circ$.

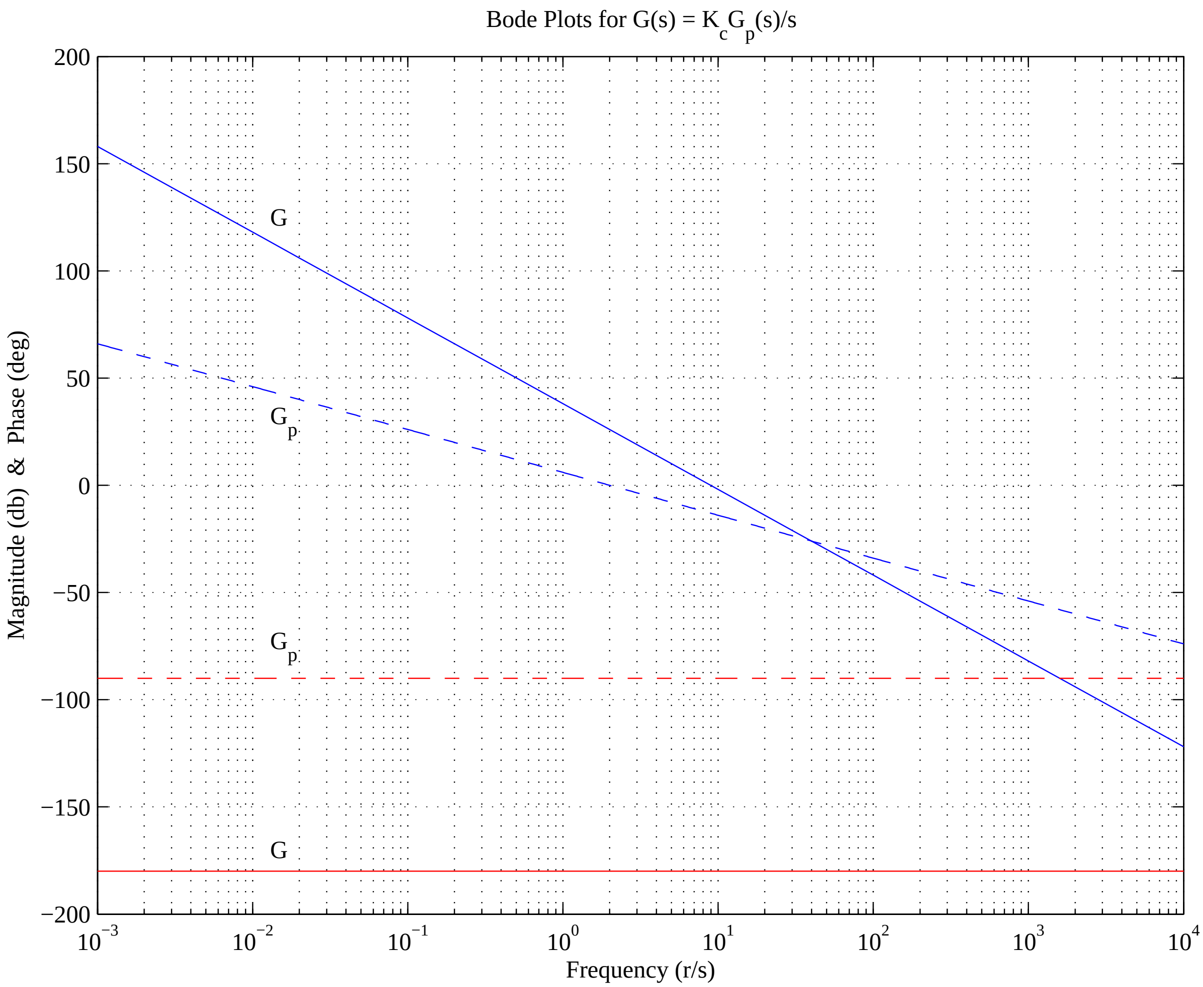

Fig. 2. Bode plots after the steady-state error specification has been satisfied.

F. Determination of $\phi_{\max}$ *and* α_d

Given the value of the uncompensated phase margin from the previous step, we can now determine the amount of positive phase shift that the lead compensator must provide. The compensator must move the phase curve of $G(j\omega)$ at $\omega = \omega_x = 5$ r/s upward from its current value of $-180°$ to the value needed to satisfy the phase margin specification. A safety factor of $10°$ will be included in this phase shift. Thus, the amount of phase shift that the lead compensator needs to provide at $\omega = \omega_x$ is

$$\begin{aligned} \phi_{\max} &= PM_{specified} + 10° - PM_{uncompensated} \\ &= 45° + 10° - 0° = 55° \end{aligned} \tag{8}$$

Knowing $\phi_{\max}$, we can compute the value of α_d. From "*Phase Lead Compensator Design Using Bode Plots*", the largest angle produced by the compensator is

$$\sin(\phi_{\max}) = \frac{1-\alpha_d}{1+\alpha_d} \tag{9}$$

so the value of α_d is computed from

$$\begin{aligned} \alpha_d &= \frac{1-\sin(55°)}{1+\sin(55°)} \\ &= 0.099 \end{aligned} \tag{10}$$

and the compensator's pole–zero combination will be related by the ratio $z_{cd}/p_{cd} = \alpha_d = 0.099$.

The value of $\phi_{\max} = 55°$ in this example is at (or near) the upper limit for a single stage of lead compensation. Many references state that $\alpha_d \geq 0.1$ should be used for the lead compensator to prevent excessively large component values and to limit the amount of undesired shift in the magnitude curve of $G(s)$ due to the compensator. The value $\alpha_d = 0.1$ corresponds to a maximum phase shift $\phi_{\max} \approx 55°$. We will assume in this example that $\alpha_d = 0.099$ is acceptable for a single stage of lead compensation.

G. Determination of z_{cd} and p_{cd}

The next and last step in the design of the transfer function for the lead compensator is to determine the values of the pole and zero. Note that in the design of the lead part of the lag-lead compensator, we did not have to determine the new gain crossover frequency. That frequency is specified, and the lag part of the compensator will take care of the shift in the magnitude curve caused by the lead part of the compensator.

Knowing the value of α_d and the specified value of $\omega_{x-compensated}$, there are no decisions to be made at this point in the design. Only simple calculations are needed to compute z_{cd} and p_{cd}.

As mentioned in Section II-A, the frequency $\omega_{\max}$ is the geometric mean of z_{cd} and p_{cd}; that is, $\omega_{\max} = \sqrt{z_{cd}p_{cd}}$. Since $\omega_{\max} = \omega_{x-compensated}$ by design, the compensator's zero and pole are computed from

$$\begin{aligned} z_{cd} &= \omega_{x-compensated}\sqrt{\alpha_d}, \qquad p_{cd} = \frac{z_{cd}}{\alpha_d} \\ z_{cd} &= 5\sqrt{0.099} = 1.58, \qquad p_{cd} = \frac{1.58}{0.099} = 15.9 \end{aligned} \tag{11}$$

Including that part of the compensator needed to satisfy the steady-state error, the transfer function for the compensator at this stage of the design is

$$\begin{aligned} G_{c\_lead}(s) &= \frac{40\,(s/1.58+1)}{s\,(s/15.9+1)} = \frac{40\,(0.634s+1)}{s\,(0.063s+1)} \\ &= \frac{402.4\,(s+1.58)}{s\,(s+15.9)} \end{aligned} \tag{12}$$

Figure 3 shows the Bode plots for the series combination $G_{c\_lead}(s)G_p(s)$. The dashed lines are the magnitude and phase for $K_cG_p(s)/s$. The phase curve of the lead-compensated system is seen to have the correct value at $\omega = 5$ r/s to satisfy the phase margin specification. The magnitude has been shifted upward from that of $K_cG_p(s)/s$. The purpose of the lag part of the compensator is to drop the magnitude of $G_{c\_lead}(s)G_p(s)$ down to 0 db at the specified gain crossover frequency.

H. Determination of α_g

Once the lead part of the compensator has been determined, the amount of magnitude attenuation that the lag compensator must provide at the specified gain crossover frequency can be evaluated. For all frequencies greater than about $4 * z_{cg}$, the magnitude $|(j\omega/z_{cg}+1)\,/\,(j\omega/p_{cg}+1)|$ of the lag compensator is $-20*\log_{10}(\alpha_g)$ db. Therefore, high frequencies (relative to z_{cg}) will be attenuated by $-20*\log_{10}(\alpha_g)$ db. Thus, α_g can be determined by evaluating the magnitude $|G_{c\_lead}(j\omega)G_p(j\omega)|$ at the specified $\omega_{x\_compensated}$.

If the Bode plots of $G_{c\_lead}(j\omega)G_p(j\omega)$ are available, then this magnitude can be read directly from the graph at the specified gain crossover frequency. If only the plots of $G(j\omega)$ defined in (6) are available, then the following relation can be used to compute $|G_{c\_lead}(j\omega)G_p(j\omega)|$ at $\omega = \omega_{x\_compensated}$.

$$|G_{c\_lead}(j\omega_{x\_compensated})G_p(j\omega_{x\_compensated})|_{db} = |G\,(j\omega_{x\_compensated})|_{db} + 10\log_{10}\left(\frac{1}{\alpha_d}\right) \tag{13}$$

For our example,

$$\begin{aligned} |G_{c\_lead}(j\omega_{x\_compensated})G_p(j\omega_{x\_compensated})|_{db} &= 20\log_{10}\left(\frac{80}{5^2}\right) + 10\log_{10}\left(\frac{1}{0.099}\right) \\ &= 10.10_{db} + 10.03_{db} \\ &= 20.13_{db} \end{aligned} \tag{14}$$

and

$$\begin{aligned} \alpha_g &= 10^{\left(\frac{20.13}{20}\right)} \\ &= 10.15 \end{aligned} \tag{15}$$

This value of α_g will provide the necessary attenuation at $\omega = 5$ r/s to establish that frequency as the compensated gain crossover frequency. This value of α_g is slightly above the upper limit generally used for a lag compensator. We will assume that is an acceptable value.

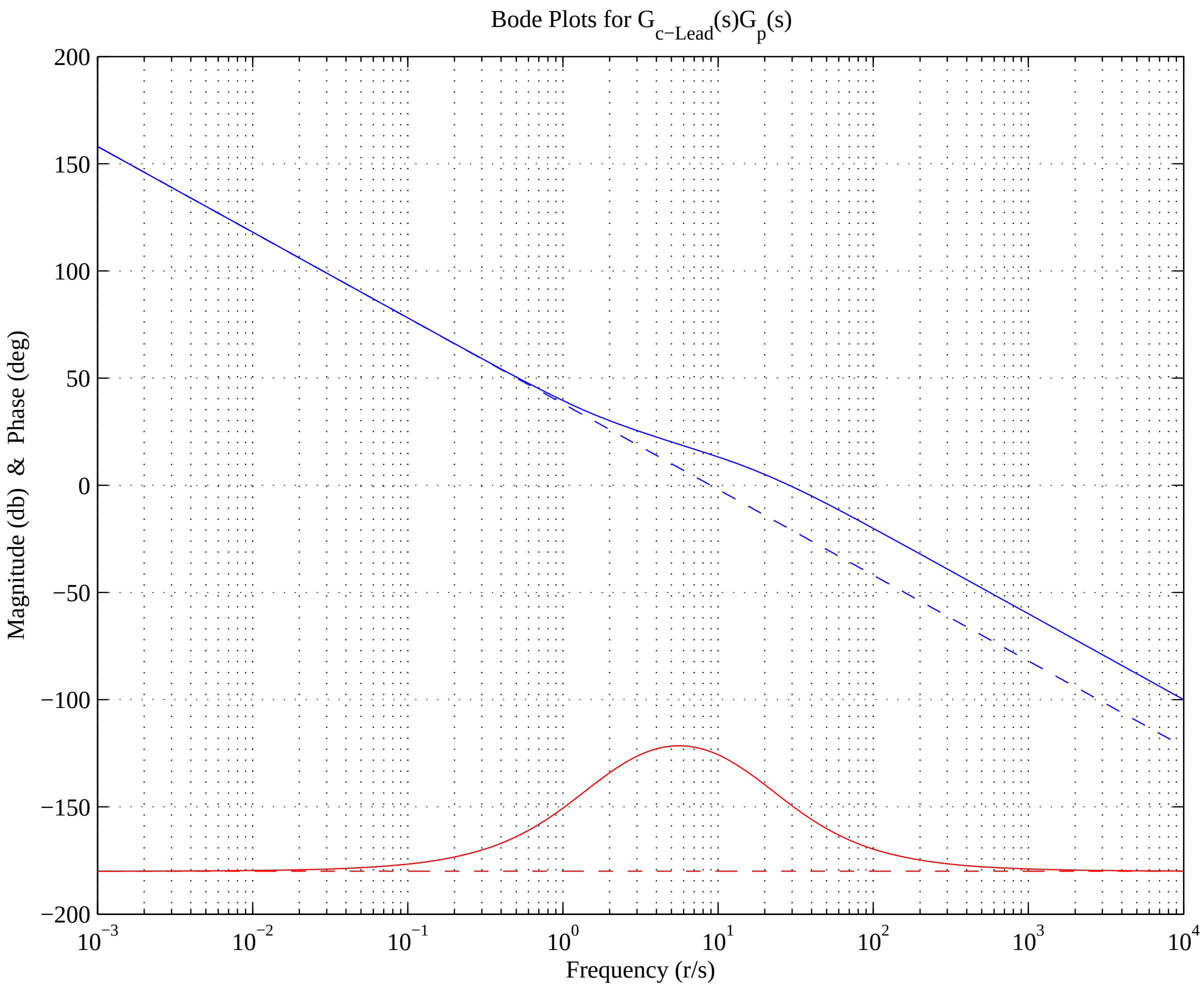

Fig. 3. Bode plots after the lead portion of the lag-lead compensator has been designed.

I. Determination of z_{cg} and p_{cg}

The last step in the design of the transfer function for the lag-lead compensator is to determine the values of the pole and zero for the lag portion of the compensator. We have already determined their ratio α_g, so only one of those terms is a free variable. We will choose to place the compensator zero and then compute the pole location from $p_{cg} = z_{cg}/\alpha_g$. We will use the same approach as described in "*Phase Lag Compensator Design Using Bode Plots*", namely placing the compensator zero z_{cg} one decade lower in frequency than the compensated gain crossover frequency. This assumes that only one stage of lag compensator will be used. The general rule of thumb if n_{stage} stages of lag compensator are being used is

$$z_{cg} = \frac{\omega_{x\_compensated}}{10 n_{stage}} \tag{16}$$

In this example the lag compensator's zero and pole are located at

$$z_{cg} = \frac{5}{10} = 0.5, \qquad p_{cg} = \frac{z_{cg}}{\alpha_g} = \frac{0.5}{10.15} = 0.049 \tag{17}$$

and the final transfer function for the lag-lead compensator is

$$\begin{aligned} G_{c\_lag\_lead} &= \frac{40\,(s/1.58+1)\,(s/0.5+1)}{s\,(s/15.9+1)\,(s/0.049+1)} \\ &= \frac{40\,(0.634s+1)\,(2s+1)}{s\,(0.063s+1)\,(20.3s+1)} \\ &= \frac{39.6\,(s+1.58)\,(s+0.5)}{s\,(s+15.9)\,(s+0.049)} \end{aligned} \tag{18}$$

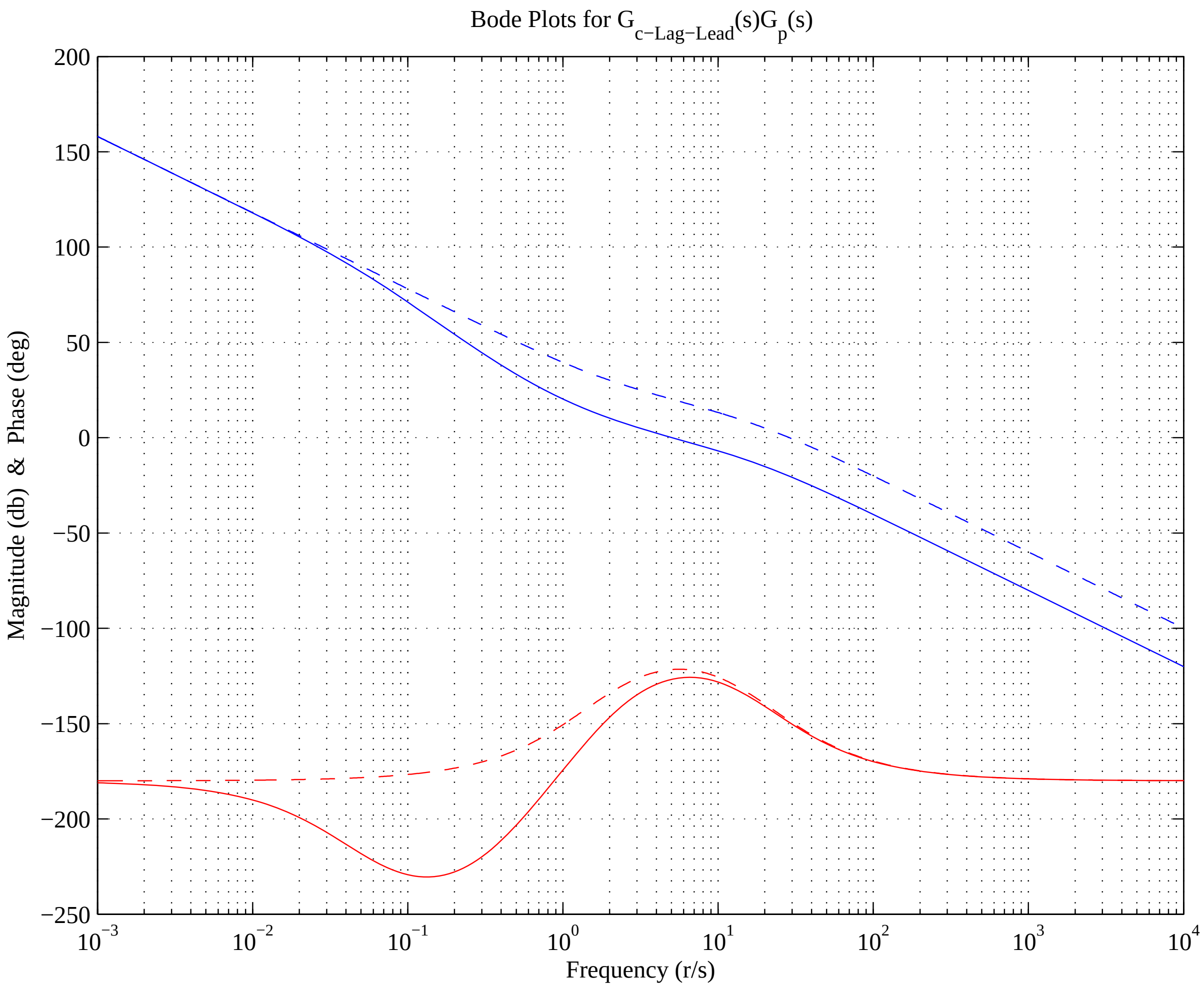

Fig. 4. Bode plots for the lag-lead compenated system.

The Bode plots for the final lag-lead compensated system are shown in Fig. 4. The gain crossover frequency is 5 r/s and the phase margin is $49.9°$, so both of those specifications have been satisfied. The error constant for the compensated system is

$$\begin{aligned} K_x &= \lim_{s \to 0} \left[s^2 \cdot \frac{39.6\,(s+1.58)\,(s+0.5)}{s\,(s+15.9)\,(s+0.049)} \cdot \frac{2}{s} \right] = 80 \\ &= \lim_{s \to 0} \left[s^2 \cdot \frac{40\,(s/1.58+1)\,(s/0.5+1)}{s\,(s/15.9+1)\,(s/0.049+1)} \cdot \frac{2}{s} \right] = 80 \end{aligned} \tag{19}$$

so the steady-state error is $e_{ss} = 1/80 = 0.0125$, and that specification has also been satisfied. Therefore, the compensator in (18) is one that allows all the specifications to be satisfied for the given plant.

III. Design Example

A. *Plant and Specifications*

The plant to be controlled is described by the transfer function

$$\begin{aligned} G_p(s) &= \frac{280\,(s+0.5)}{s\,(s+0.2)\,(s+5)\,(s+70)} \\ &= \frac{2\,(s/0.5+1)}{s\,(s/0.2+1)\,(s/5+1)\,(s/70+1)} \end{aligned} \tag{20}$$

This is a Type 1 system, so the closed-loop system will have zero steady-state error for a step input, and a non-zero, finite steady-state error for a ramp input (assuming that the closed-loop system is stable). As shown in the next section, the error constant for a ramp input is $K_{x-plant} = 2$. At low frequencies, the plant has a magnitude slope of -20 db/decade, and at high frequencies the slope is -60 db/decade. The phase curve starts at $-90°$ and ends at $-270°$.

The specifications that must be satisfied by the closed-loop system are:

- steady-state error for a ramp input $e_{ss\_specified} \leq 0.02$;
- phase margin $PM_{specified} \geq 55°$;
- gain crossover frequency $\omega_{x\_specified} = 5$ r/s.

When compared with the design examples in "*Phase Lag Compensator Design Using Bode Plots*" and "*Phase Lead Compensator Design Using Bode Plots*", it can be seen that the specification on the gain crossover frequency lies between the final values achieved in those two designs. Therefore, neither lag compensation nor lead compensation will work by themselves. Lag-lead compensation is needed, and the following paragraphs will illustrate how the procedure is applied to design the compensator for this system that will allow the specifications to be satisfied.

B. Compensator Gain

The given plant is Type 1, and the steady-state error specification is for a ramp input, so the compensator does not need to have any poles at $s = 0$. Only the gain K_c needs to be computed for steady-state error. The steady-state error for a ramp input for the given plant is

$$\begin{aligned} K_{x\_plant} &= \lim_{s \to 0} \left[s \cdot \frac{280\,(s+0.5)}{s\,(s+0.2)\,(s+5)\,(s+70)} \right] \qquad (21) \\ &= \lim_{s \to 0} \left[s \cdot \frac{2\,(s/0.5+1)}{s\,(s/0.2+1)\,(s/5+1)\,(s/70+1)} \right] \\ &= 2 \\ e_{ss\_plant} &= \frac{1}{K_x} = 0.5 \qquad (22) \end{aligned}$$

Since the specified value of the steady-state error is 0.02, the required error constant is $K_{x\_required} = 50$. Therefore, the compensator gain is

$$\begin{aligned} K_c &= \frac{e_{ss\_plant}}{e_{ss\_specified}} = \frac{0.5}{0.02} = 25 \qquad (23) \\ &= \frac{K_{x\_required}}{K_{x\_plant}} = \frac{50}{2} = 25 \end{aligned}$$

This value for K_c will satisfy the steady-state error specification, and the rest of the compensator design will focus on the phase margin and gain crossover frequency specifications.

C. The Bode Plots

The magnitude and phase plots for $K_c G_p(s)$ are shown in Fig. 5. The dashed magnitude curve is for $G_p(s)$ and illustrates the effect that K_c has on the magnitude. Specifically, $|K_c G_p\,(j\omega)|$ is $20\log_{10}|25| \approx 28$ db above the curve for $|G_p\,(j\omega)|$ at all frequencies. The phase curve is unchanged when the steady-state error specification is satisfied since the compensator does not have any poles at the origin. The horizontal and vertical dashed lines in the figure indicate the specified gain crossover frequency of 5 r/s and the phase shift of $-142.5°$ at that frequency.

Note that the gain crossover frequency of $K_c G_p(s)$ is larger than that for $G_p(s)$; the crossover frequency has moved to the right in the graph. The closed-loop bandwidth will have increased in a similar manner. The phase margin has decreased due to K_c, so satisfying the steady-state error has made the system less stable; in fact, increasing K_c in order to decrease the steady-state error can even make the closed-loop system unstable. Maintaining stability and achieving the desired phase margin is the task of the pole–zero combination in the compensator.

Our ability to graphically make the various measurements needed during the design obviously depends on the accuracy and resolution of the Bode plots of $|K_c G_p\,(j\omega)|$. High resolution plots like those obtained from MATLAB allow us to obtain reasonably accurate measurements. Rough, hand-drawn sketches would yield much less accurate results and might be used only for first approximations to the design. Being able to access the actual numerical data allows for even more accurate results than the MATLAB-generated plots. The procedure that I use when working in MATLAB generates the data arrays for frequency, magnitude, and phase from instructions such as the following:

```
w = logspace(N1,N2,1+100*(N2-N1));
[mag,ph] = bode(num,den,w);
semilogx(w,20*log10(mag),w,ph),grid
```

where `N1`$= \log_{10}(\omega_{\min})$, `N2` $= \log_{10}(\omega_{\max})$, and `num`, `den` are the numerator and denominator polynomials, respectively, of $K_c G_p\,(s)$. For this example,

`N1`$= -3$;

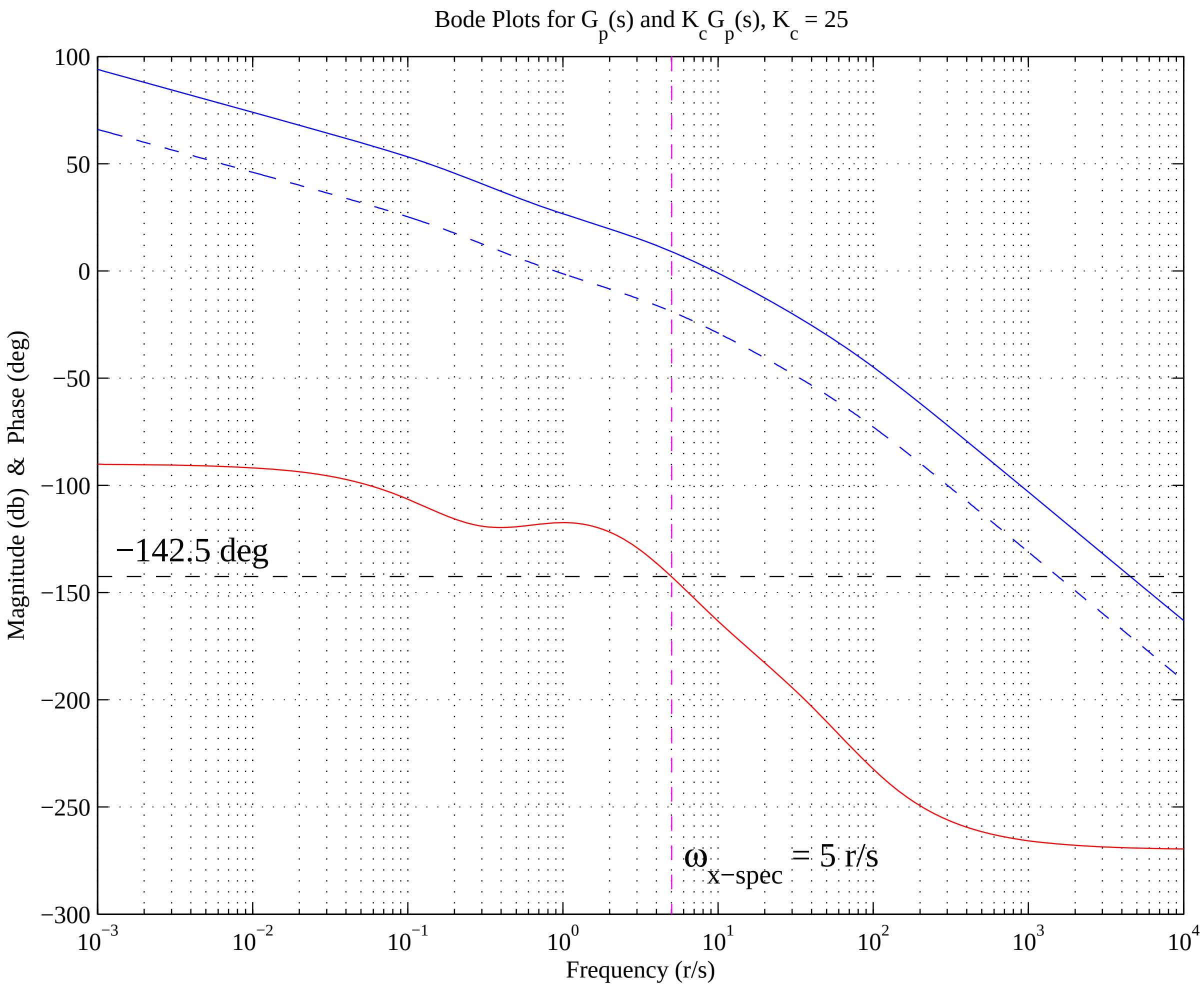

Fig. 5. Bode plots for the plant after the steady-state error specification has been satisfied.

```
N2= 4;
num= 25 * 280 * [ 1   0.5 ];
den=conv([ 1   0.2   0 ], conv([ 1   5 ], [ 1   70 ]));
```

The data arrays `mag`, `ph`, and `w` can be searched to obtain the various values needed during the design of the compensator.

D. Uncompensated Phase Margin

The specified value for the compensated gain crossover frequency is $\omega_{x-compensated} = 5$ r/s. The phase shift of the plant at that frequency is $\angle K_c G_p\left(j\omega_{x-compensated}\right) = -142.5°$. Although this frequency is not the uncompensated gain crossover frequency, we will define the uncompensated phase margin in the usual way at that frequency. Therefore, the uncompensated phase margin is

$$PM_{uncompensated} = 180° + (-142.5°) = 37.5° \tag{24}$$

Since the uncompensated phase margin is positive, the closed-loop system formed by placing unity feedback around $K_c G_p\left(s\right)$ is stable, but the phase margin is smaller than the specified value, so compensation is required.

E. Determination of $\phi_{\max}$ and α_d

The lead part of the compensator will need to move the phase curve up at the specified gain crossover frequency by an amount

$$\phi_{\max} = 55° + 10° - 37.5° = 27.5° \tag{25}$$

Since this value of $\phi_{\max}$ is well below the limit of $55°$, we can design a single-stage lead compensator. That will provide the correct phase margin for the compensated system. The value of α_d that corresponds to this $\phi_{\max}$ is

$$\alpha_d = \frac{1 - \sin\left(27.5°\right)}{1 + \sin\left(27.5°\right)} = 0.368 \tag{26}$$

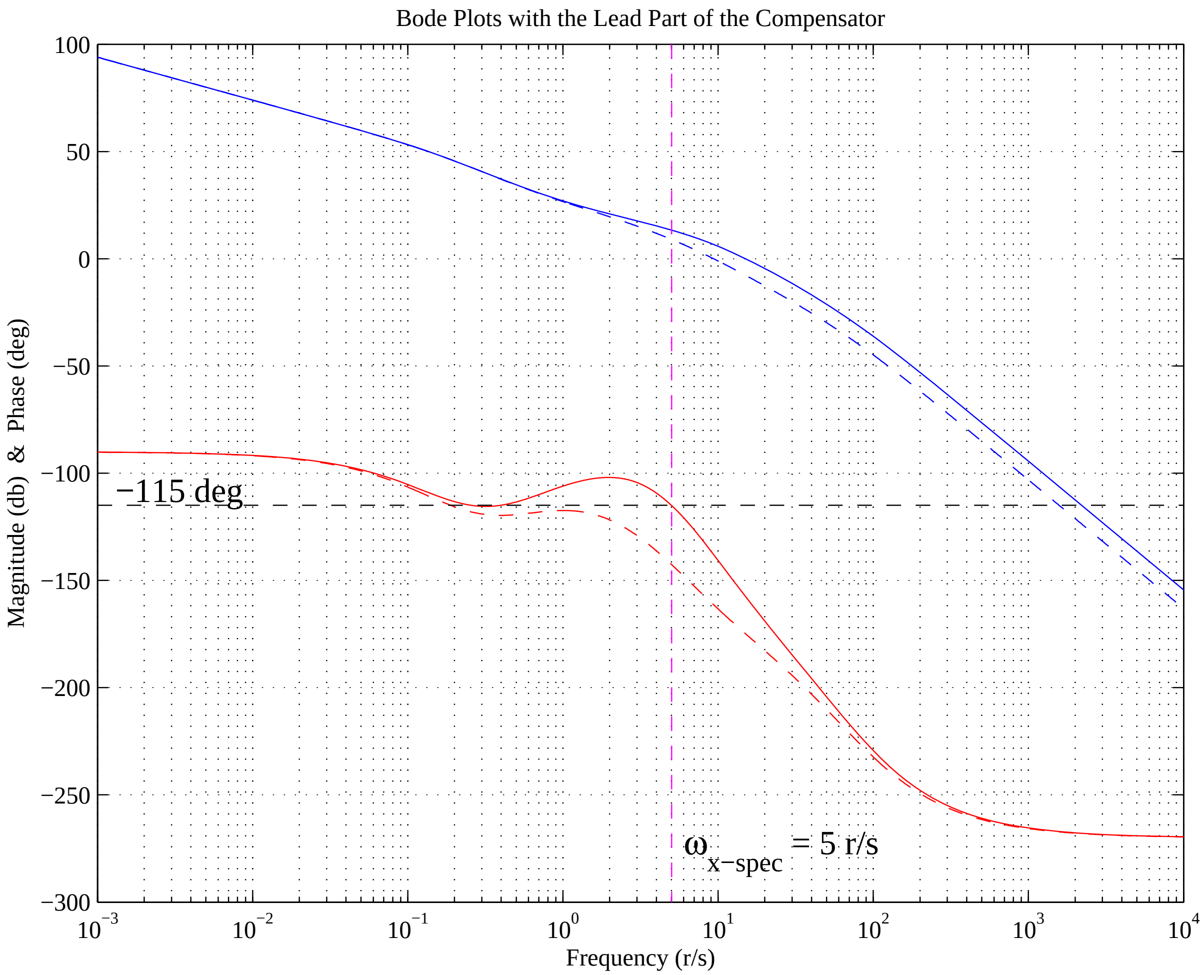

Fig. 6. Bode plots after the design of the lead part of the lag-lead compensator.

F. *Lead Compensator Zero and Pole*

Now that we have a specified value for $\omega_{x\_compensated}$ and the value of α_d, we can determine the values for the zero and pole for the lead part of the compensator from (11). These values are

$$
\begin{aligned}
z_{cd} &= 5\sqrt{0.368} = 3.03 \\
p_{cd} &= \frac{3.03}{0.368} = 8.24
\end{aligned}
\tag{27}
$$

The lead part of the lag-lead compensator (including K_c) for this example is

$$
\begin{aligned}
G_{c\_lead}(s) &= \frac{25\,(s/3.03+1)}{(s/8.24+1)} = \frac{25\,(0.33s+1)}{(0.121+1)} \\
&= \frac{67.9\,(s+3.03)}{(s+8.24)}
\end{aligned}
\tag{28}
$$

Figure 6 shows the Bode plots for the system after the lead part of the compensator has been designed. The phase shift is seen to have the correct value at 5 r/s to satisfy the phase margin specification with 10° of safety factor. The magnitude curve has been shifted upward by the phase lead part of the compensator by the amount $10\log_{10}(1/\alpha_d)$. The lag part of the compensator will need to drop this combined magnitude value down to 0 db at $\omega = 5$ r/s.

G. *Determination of α_g*

Now that the lead part of the compensator has been designed, we know the amount of magnitude shift that will occur at $\omega = 5$ r/s. We can measure the magnitude $|K_c G_p(j5)|_{db}$ from the Bode plots or search the data arrays for the magnitude value at the corresponding frequency index. We can compute the amount of magnitude shift caused by the lead compensator from $10\log_{10}(1/\alpha_d)$. The amount of magnitude attenuation that is needed at $\omega = 5$ r/s is

$$\begin{aligned} \left|G_{c\_lead}(j5)\,G_p(j5)\right|_{db} &= \left|G_p(j5)\right|_{db} + 10\log_{10}(1/\alpha_d) \\ &= 9.05 + 4.34 \\ &= 13.39\ db \end{aligned} \tag{29}$$

The lag part of the compensator must provide this amount of attenuation in order to make $\omega = 5$ r/s actually be the gain crossover frequency. The corresponding value of α_g needed for this attenuation is the magnitude of 13.39 db converted to absolute value. Thus,

$$\alpha_g = 10^{\left(\dfrac{13.39}{20}\right)} = 4.67 \tag{30}$$

H. Determination of z_{cg} and p_{cg}

The value for the zero of the lag part of the compensator will correspond to one decade in frequency below the specified gain crossover frequency as usual. The lag compensator's zero and pole are

$$\begin{aligned} z_{cg} &= \frac{\omega_{x\_specified}}{10} = \frac{5}{10} = 0.5 \\ p_{cg} &= \frac{z_{cg}}{\alpha_g} = \frac{0.5}{4.67} = 0.107 \end{aligned} \tag{31}$$

and the final transfer function for the lag-lead compensator is

$$\begin{aligned} G_{c\_lag\_lead}(s) &= \frac{25\,(s/3.03+1)\,(s/0.5+1)}{(s/8.24+1)\,(s/0.107+1)} = \frac{25\,(0.33s+1)\,(2s+1)}{(0.121+1)\,(9.34s+1)} \\ &= \frac{14.5\,(s+3.03)\,(s+0.5)}{(s+8.24)\,(s+0.107)} \end{aligned} \tag{32}$$

The closed-loop system consisting of this compensator in series with $G_p(s)$ in (20) in a unity-feedback configuration will satisfy all of the specifications. The Bode plots for the final compensated system are shown in Fig. 7, and the closed-loop step responses for the various systems are shown in Fig. 8. The Bode plots clearly show that the gain crossover frequency and phase margin specifications have been satisfied. The step response for the lead-compensated system is the curve with peak overshoot nearly as large as the response from $K_cG_p(s)$. The step response of the lag-lead-compensated system has much smaller overshoot than $K_cG_p(s)$, but a slightly longer settling time. Compared with $G_p(s)$, the final compensated system has a shorter settling time, slightly less overshoot, approximately the same phase margin, and much smaller steady-state error.

I. Implementation of the Compensator

Ogata [3] presents a table showing analog circuit implementations for various types of compensators. The circuit for the lag-lead compensator is the series combination of two inverting operational amplifiers. The first amplifier has an input impedance that has resistor R_1 in series with capacitor C_1, with that series combination in parallel with resistor R_3. The feedback impedance has resistor R_2 in series with capacitor C_2, with that series combination in parallel with resistor R_4. The second amplifier has input and feedback resistors R_5 and R_6, respectively.

Assuming that the op amps are ideal, the transfer function for this circuit is

$$\begin{aligned} \frac{V_{out}(s)}{V_{in}(s)} &= \frac{R_6R_4}{R_5R_3}\cdot\frac{[s\,(R_1+R_3)\,C_1+1]}{(sR_1C_1+1)}\cdot\frac{(sR_2C_2+1)}{[s\,(R_2+R_4)\,C_2+1]} \\ &= \frac{R_6R_4}{R_5R_3}\cdot\frac{R_2\,(R_1+R_3)}{R_1\,(R_2+R_4)}\cdot\frac{\left(s+\dfrac{1}{(R_1+R_3)\,C_1}\right)}{\left(s+\dfrac{1}{R_1C_1}\right)}\cdot\frac{\left(s+\dfrac{1}{R_2C_2}\right)}{\left(s+\dfrac{1}{(R_2+R_4)\,C_2}\right)} \end{aligned} \tag{33}$$

Comparing (33) with $G_{c\_lag\_lead}(s)$ in (1) shows that the following relationships hold:

$$\begin{aligned} &K_c = \frac{R_6R_4}{R_5R_3}, \qquad z_{cd} = 1/\left(R_1+R_3\right)C_1, \qquad p_{cd} = 1/R_1C_1, \qquad \alpha_d = \frac{R_1}{R_1+R_3} \\ &z_{cg} = 1/R_2C_2, \qquad p_{cg} = 1/\left(R_2+R_4\right)C_2, \qquad \alpha_g = \frac{R_2+R_4}{R_2} \end{aligned} \tag{34}$$

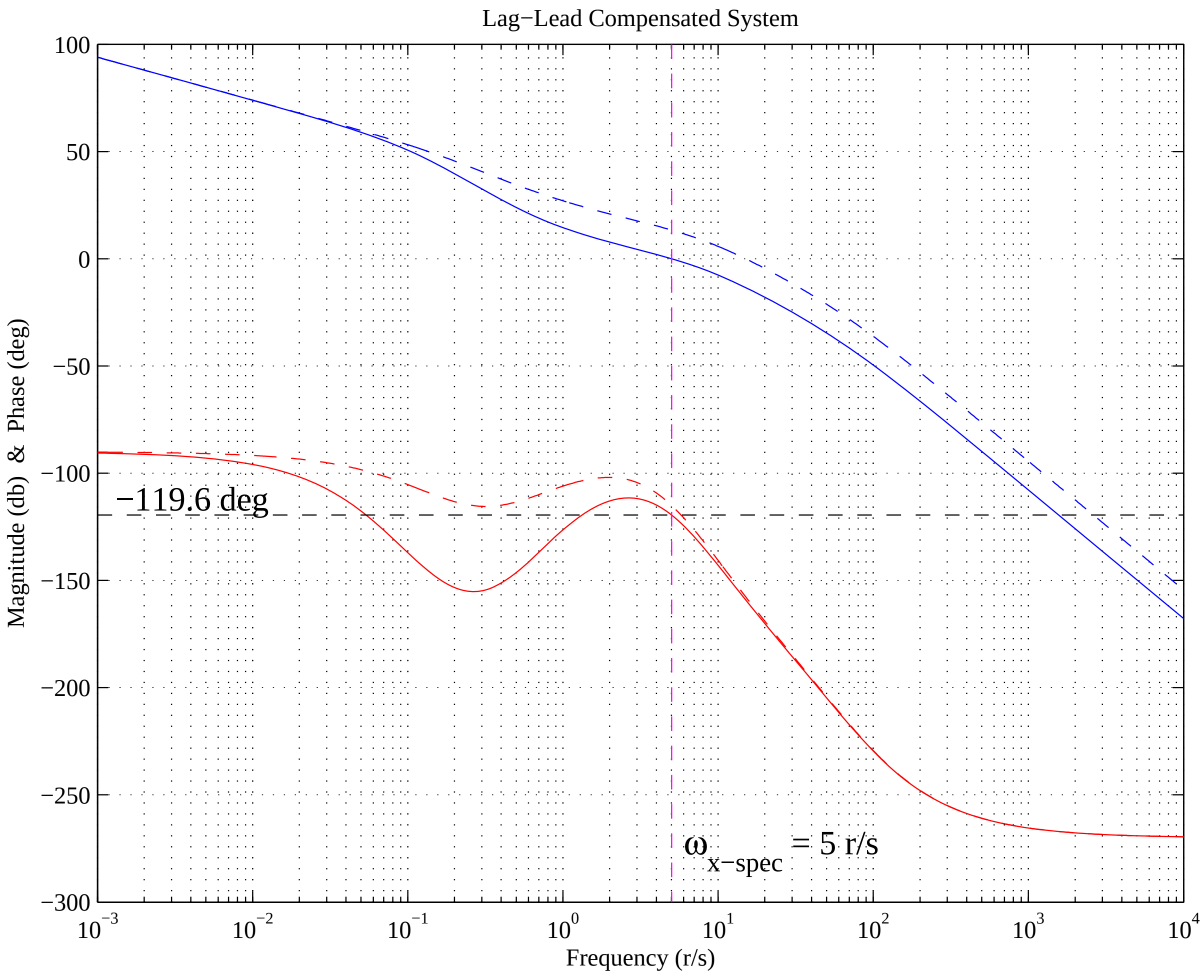

Fig. 7. Bode plots for the final lag-lead compensated system.

To implement the compensator using the circuit in [3], note that there are 8 unknown circuit elements (R_1, C_1, R_2, C_2, R_3, R_4, R_5, R_6) and 5 compensator parameters (K_c, z_{cd}, p_{cd}, z_{cg}, p_{cg}). Therefore, three of the circuit elements can be chosen to have convenient values. To implement the final lag-lead compensator $G_{c\_lag\_lead}(s)$, we can use the following values:

$$\begin{aligned}
C_1 &= C_2 = 0.1\ \mu\text{F}\ = 10^{-7}\ \text{F}, \qquad R_5 = 10\ \text{K}\Omega = 10^4\ \Omega \\
R_1 &= \frac{1}{p_{cd}C_1} = 1.21\ \text{M}\Omega = 1.21 \cdot 10^6\ \Omega \\
R_3 &= \frac{R_1\,(1-\alpha_d)}{\alpha_d} = 2.08\ \text{M}\Omega = 2.08 \cdot 10^6\ \Omega \\
R_2 &= \frac{1}{z_{cg}C_2} = 20\ \text{M}\Omega = 2 \cdot 10^7\ \Omega \\
R_4 &= R_2\,(\alpha_g - 1) = 73.4\ \text{M}\Omega = 7.34 \cdot 10^7\ \Omega \\
R_6 &= \frac{K_c R_3 R_5}{R_4} = 7.09\ \text{K}\Omega = 7.09 \cdot 10^3\ \Omega
\end{aligned} \tag{35}$$

where the elements in the first row of (35) were specified and the remaining elements were computed from (34).

J. Summary

In this example, the lag-lead compensator in (32) is able to satisfy all three of the specifications imposed on the system given in (20). In addition to satisfying the phase margin, crossover frequency, and steady-state error specifications, the lag-lead compensator also produced a step response with a shorter settling time and less overshoot.

In summary, lag-lead compensation can provide steady-state accuracy and necessary phase margin at a specified frequency when the Bode phase plot can be moved up the necessary amount at that frequency. The philosophy of the lag-lead compensator

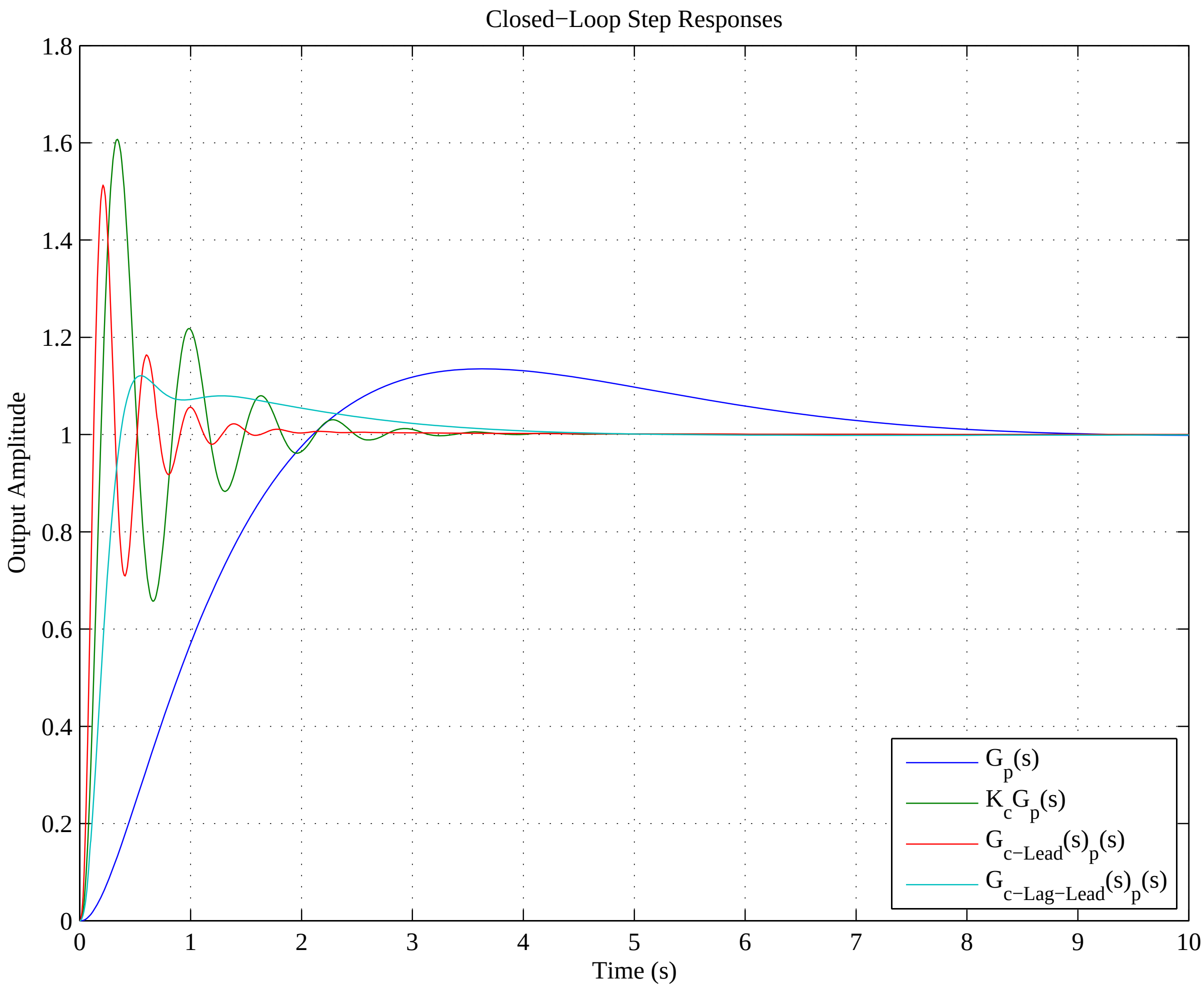

Fig. 8. Closed-loop step responses for the various systems defined in the example.

is to add positive phase shift at the specified gain crossover frequency to satisfy the phase margin specification and then to provide sufficient magnitude attenuation to satisfy the crossover frequency specification.

The following table provides a comparison between the systems in this example.

Characteristic	Symbol	$G_p(s)$	$K_cG_p(s)$	$G_{c\_lead}(s)G_p(s)$	$G_{c\_lag\_lead}(s)G_p(s)$
steady-state error	e_{ss}	0.5	0.02	0.02	0.02
phase margin	PM	62.5°	18.7°	22.5°	60.4°
gain xover freq	ω_x	0.88 r/s	9.4 r/s	15.0 r/s	5.02 r/s
time delay	T_d	1.24 sec	0.035 sec	0.026 sec	0.21 sec
gain margin	GM	87.7	3.51	2.93	13.1
gain margin (db)	GM_{db}	38.9 db	10.9 db	9.34 db	22.3 db
phase xover freq	ω_ϕ	18.1 r/s	18.1 r/s	26.5 r/s	25.9 r/s
bandwidth	ω_B	1.29 r/s	14.9 r/s	24.0 r/s	8.78 r/s
percent overshoot	PO	13.5%	60.7%	51.3%	12.1%
settling time	T_s	7.42 sec	2.39 sec	1.43 sec	3.15 sec

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