Filon's Integration Formula[3]
25.4.47[1]

$$\int_{x_0}^{x_{2n}} f(x)\cos tx\,dx = h\Big[\alpha(th)(f_{2n}\sin tx_{2n}-f_0\sin tx_0)+\beta(th)C_{2n}+\gamma(th)C_{2n-1}+\frac{2}{45}th^4S'_{2n-1}\Big]-R_n$$

25.4.48

$$C_{2n}=\sum_{i=0}^{n} f_{2i}\cos(tx_{2i})-\frac{1}{2}[f_{2n}\cos tx_{2n}+f_0\cos tx_0]$$

25.4.49

$$C_{2n-1}=\sum_{i=1}^{n} f_{2i-1}\cos tx_{2i-1}$$

25.4.50

$$S'_{2n-1}=\sum_{i=1}^{n} f^{(3)}_{2i-1}\sin tx_{2i-1}$$

25.4.51

$$R_n=\frac{1}{90}nh^5f^{(4)}(\xi)+O(th^7)$$

25.4.52

$$\begin{aligned}
\alpha(\theta) &= \tfrac{1}{\theta}+\tfrac{\sin 2\theta}{2\theta^2}-\tfrac{2\sin^2\theta}{\theta^3}\\
\beta(\theta) &= 2\left(\tfrac{1+\cos^2\theta}{\theta^2}-\tfrac{\sin 2\theta}{\theta^3}\right)\\
\gamma(\theta) &= 4\left(\tfrac{\sin\theta}{\theta^3}-\tfrac{\cos\theta}{\theta^2}\right)
\end{aligned}$$

For small θ we have
25.4.53

$$\alpha = \tfrac{2\theta^3}{45}-\tfrac{2\theta^5}{315}+\tfrac{2\theta^7}{4725}-\cdots$$

$$\begin{aligned}\beta &= \tfrac{2}{3} + \tfrac{2\theta^2}{15} - \tfrac{4\theta^4}{105} + \tfrac{2\theta^6}{567} - \cdots \\ \gamma &= \tfrac{4}{3} - \tfrac{2\theta^2}{15} + \tfrac{\theta^4}{210} - \tfrac{\theta^6}{11340} + \cdots\end{aligned}$$

25.4.54

$$\int_{x_0}^{x_{2n}} f(x) \sin tx \, dx = h\Big[\alpha(th)(f_0 \cos tx_0 - f_{2n} \cos tx_{2n}) + \beta S_{2n} + \gamma S_{2n-1} + \frac{2}{45} th^4 C'_{2n-1}\Big] - R_n$$

25.4.55

$$S_{2n} = \sum_{i=0}^{n} f_{2i} \sin(tx_{2i}) - \frac{1}{2}[f_{2n} \sin(tx_{2n}) + f_0 \sin(tx_0)]$$

25.4.56

$$S_{2n-1} = \sum_{i=1}^{n} f_{2i-1} \sin(tx_{2i-1})$$

25.4.57

$$C'_{2n-1} = \sum_{i=1}^{n} f^{(3)}_{2i-1} \cos(tx_{2i-1})$$