Designing Implodable Underwater Systems to Minimize Implosion Pulse Severity

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Abstract*—Underwater implosion is the rapid and catastrophic collapse of a pressure vessel subjected to external pressure, resulting in a very short but high-pressure pulse in the surrounding water that can potentially damage adjacent structures or personnel. Any system with an internal pressure below external pressure is susceptible to implosion. The pressure pulse from a single implosion has been known to trigger subsequent sympathetic implosions. This work investigates methods of reducing the implosion pulse severity for unstiffened metallic cylindrical pressure vessels. The implosion pulse energy is proportional to the maximum system kinetic energy developed during collapse. It can be reduced by (1) increasing the plastic energy dissipated by the collapsing structure, or (2) increasing the energy required to compress the internal gas. Plastic energy dissipation is increased by triggering higher buckling modes through introduction of geometric imperfections. Numerical simulations show that this technique can reduce implosion pulse energy for a sample cylinder by up to 33%, while reducing the buckling strength of the cylinder by only 0.5%. The energy required for gas compression can be increased by initially pressurizing the interior of the implodable. The benefit from this technique is bounded by other limitations on internal pressure, such as equipment and human survivability. A slight increase in gas compression energy is also realized by substituting a noble gas for air. These techniques can be applied, singly or in combination, to any implodable design to significantly reduce the pulse severity and minimize the risk of sympathetic implosions.

Keywords*—*Implosion, Underwater vehicles, Cylindrical shell, Fluid-structure interaction, Pressure pulse, Sympathetic implosion.

I. Introduction

Underwater implosion is the rapid and catastrophic collapse of a pressure vessel subjected to external pressure. Implosion creates a very short but high-pressure pulse in the surrounding water that can potentially damage adjacent structures or personnel (representative pulse shown in Fig. 1). An implosion pressure pulse is very similar to the pulse created by the collapse of a gas bubble from an underwater explosion. Any pressure vessel with an internal pressure below external has the potential to implode [1]. This includes submarines, unmanned underwater vehicles (UUVs), undersea pipelines, submerged air flasks, and even light bulbs.

The implosion phenomenon has been thoroughly studied and reported by Turner and Ambrico [2] and Farhat et al. [3]. Implosion is a complex, fully-coupled fluid-structure interaction problem. The fluid pressure causes the structure to deform, and the structural deformation in turn affects the fluid pressure. Much of the previous work on implosion was focused on numerical simulations to accurately predict the implosion pulse for a given metallic cylinder, and has been quite successful in doing so. Alternatively, Gish and Wierzbicki [4] developed an analytical method to estimate the energy in the implosion pulse, without reliance on computationally expensive simulations.

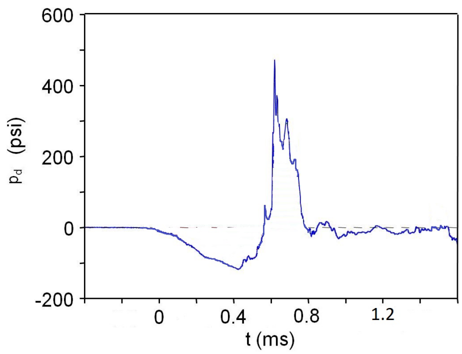


Fig. 1: Typical pressure pulse for an underwater implosion event.

The pressure pulse from a single implosion may trigger subsequent secondary implosions (i.e., cascading or sympathetic implosions). For example, at the Super-Kamiokande Cherenkov detector facility in Japan in 2001, a single failed photomultiplier tube triggered the implosion of 6777 additional tubes [5]. An implosion pulse can also be a significant hazard to personnel in the vicinity. For this reason, the U.S. Navy imposes strict safety requirements on all implodable systems involving humans. It is desirable to minimize the implosion pulse (quantified by pulse energy or peak pressure) in the event that implosion of the primary structure does occur. The objective of this work is to investigate methods of reducing the implosion pulse severity for unstiffened metallic cylindrical pressure vessels, which represent a broad range of implodable systems.

The effect of implosion on adjacent structures or personnel can be minimized in two general ways: (1) prohibit implosion from occurring, by designing the primary implodable with a very high safety factor for the worst-case operating conditions, or (2) allow that implosion may occur under certain conditions and design the implodable to minimize the pulse energy or peak pressure when it occurs, so as to reduce the likelihood of secondary implosions. The former approach is necessary in cases where implosion is unacceptable (such as manned submarines), but is extremely conservative and results in over-designed, cost-prohibitive systems for non-critical applications like UUVs. The latter approach is suitable for unmanned systems where implosion is an acceptable risk, and is the focus of this work.

II. Implosion Mechanics

A. Overview

A necessary precursor to implosion mitigation is a thorough understanding of the implosion mechanics. Implosion occurs when the critical buckling pressure of a structure is exceeded, either by hydrostatic pressure alone or by a combination of hydrostatic pressure and a triggering event like an explosion or impact. The negative dynamic pressure in Fig. 1 (from 0 to approximately 0.6 ms) is caused by the in-rush of water as the solid walls of the pressure vessel collapse. The large positive pressure spike results from the deceleration and compression of the water as the opposite walls contact each other and stop moving. The entire implosion event occurs on the order of milliseconds.

An unstiffened metallic cylinder subjected to purely hydrostatic pressure loading tends to collapse in a symmetric mode, examples of which are shown in Fig. 2. The mode is

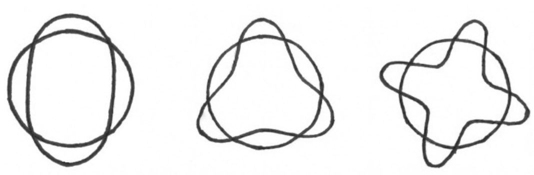


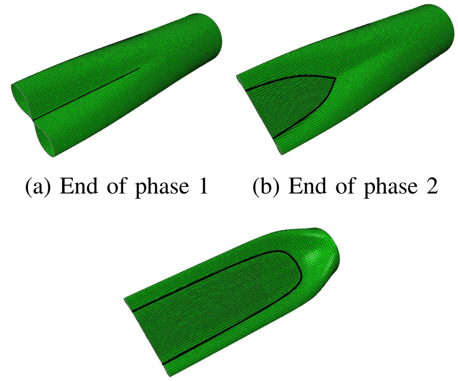
Fig. 2: Symmetric collapse modes under purely hydrostatic loading.

determined by the cylinder's length, diameter, and thickness in accordance with classical elastic-plastic buckling theory. In general, longer thinner cylinders collapse in mode 2, while shorter thicker cylinders collapse into mode 3 or 4 [6].

Regardless of the collapse mode, implosion of cylinders follows a predictable sequence of events (or phases):

- Phase 1: symmetric collapse until first contact between opposite cylinder walls at a single point on the center cross-section
- Phase 2: center cross-section continues to flatten until reaching a maximum, depending on pressure and the diameter-to-thickness ratio
- Phase 3: flattening continues longitudinally along the cylinder until reaching a final state

Figure 3 depicts the three collapse phases for a representative numerical simulation (dark lines added to highlight the flattened regions).



(a) End of phase 1 (b) End of phase 2

(c) End of phase 3

Fig. 3: Phases of mode 2 collapse (half-length shown).

B. Energy Balance

The overall energy balance during an implosion event may be expressed as:

$$KE = W - E_{int} - E_{air} \tag{1}$$

where KE is total system kinetic energy (structure plus surrounding fluid), W is the external work done on the structure by the fluid, E_{int} is the plastic energy dissipated by the structure as it collapses, and E_{air} is the energy required to compress the gas volume inside the implodable. The implosion pulse is caused by the rapid deceleration and compression of the in-rushing water at the instant when the structure's sides collapse onto themselves. The maximum value of KE occurs just after the end of phase 1, as shown in Fig. 4. Following the peak, KE decreases as some energy causes further plastic deformation (i.e., flattening) of the cylinder and the remainder is transmitted back into the fluid as the pressure pulse. It is impossible to analytically determine what portion of the kinetic energy goes into the pressure pulse, so the worst-case (100%) is assumed. Therefore, the maximum system kinetic energy developed during collapse is an upper bound on the implosion pulse energy.

Because the maximum KE occurs very near the end of phase 1, the implosion pulse energy can be closely approximated by evaluating Eq. (1) over phase 1 only. The pulse severity is reduced by reducing the maximum KE, which requires consideration of the individual terms on the right-hand side of Eq. (1).

The external work, W, is the product of the fluid pressure acting on the structure (P) and the change in volume as it collapses:

$$W = P\Delta V \tag{2}$$

This value is constant for a given implodable volume and hydrostatic pressure (i.e., operating depth), assuming complete collapse of the structure[1]. Thus, KE (and therefore pulse severity) can be reduced by increasing E_{int} and/or E_{air}.

[1]Numerical analysis indicates that the external work is also nearly independent of the collapse mode.

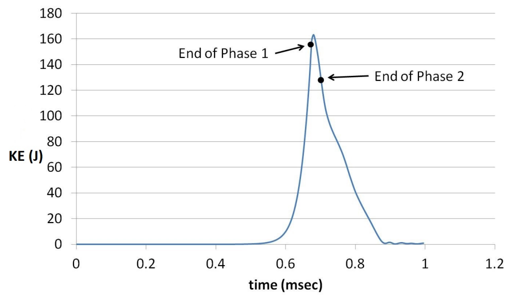


Fig. 4: Kinetic energy vs. time (from representative numerical simulation).

III. METHOD TO INCREASE E_{int}

The structural energy dissipation in an imploding cylinder was described in detail by Gish and Wierzbicki [4]. The total energy includes a bending component (due to deformation of the circular cross-section into the collapsed mode shape) and a membrane component (due to longitudinal stretching of the cylinder as it collapses). The following analytic equation was developed for the energy dissipation during phase 1 of a mode 2 collapse:

$$E_{int,phase1(mode2)} = \sigma_0 h \left(1.555hL + 1.964\frac{R^3}{L} \right) \quad (3)$$

where σ_0 is the material flow stress, h is cylinder thickness, L is the half-length, and R is the radius. The two terms in parentheses on the right-hand side of Eq. (3) represent bending and membrane energy, respectively. Figure 5 compares the analytic solution to the numerical simulation for a representative cylinder. The x-axis is the degree of collapse,

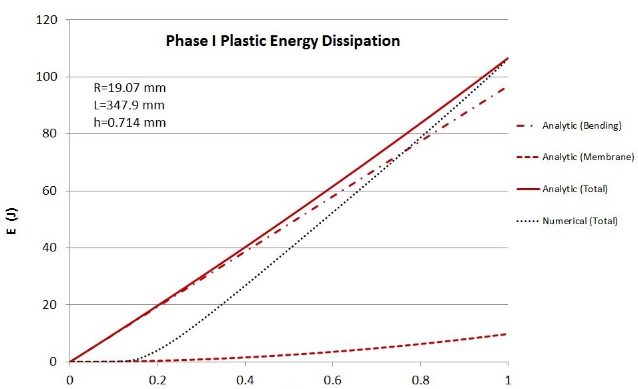


Fig. 5: Comparison of analytic and numerical energy dissipation for mode 2 collapse.

with $w_0/R = 1$ representing the instant when the opposite walls first contact each other (i.e., the end of phase 1). The figure shows very close agreement (within 1%) between the analytic and numerical solutions, and that the bending energy contribution is much larger than the membrane. It follows then that the kinematics of the collapsing cross-section (which are directly related to the mode shape) have a much larger effect on energy dissipation than do the longitudinal kinematics.

The energy dissipation can be increased for a given cylinder (i.e., without changing material or dimensions) by triggering a higher collapse mode. Classical elastic-plastic mechanics theory predicts that higher collapse modes should dissipate more energy, because of the increased circumferential bending required. For example, the intermediate mode 2 shape can be thought of as having 4 bends (plastic hinges), while the mode 3 shape has 6 and mode 4 has 8 (see Fig. 6).

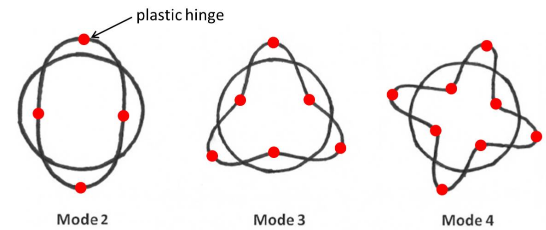


Fig. 6: Collapse modes with locations of plastic hinges.

Following the example of [4], an analytic expression for energy dissipation at the end of phase 1 for mode 3 collapse was developed:

$$E_{int,phase1(mode3)} = 3.027\sigma_0 h^2 L \quad (4)$$

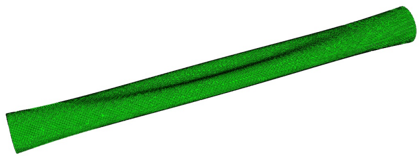
Only bending effects were included, because as shown previously the membrane effects are negligible. A summary of the development is shown in the appendix; for a full explanation of the derivation and underlying assumptions, the interested reader is referred to [4]. Equation (4) was validated with a numerical simulation and showed very good agreement (within 10%). Comparison of Eqs. (3) and (4) shows that the energy dissipation in mode 3 is expected to be nearly twice that of mode 2. This result confirms the expectation that higher modes will dissipate more energy.

Numerical simulations of Al6061-T6 cylinders subjected to hydrostatic pressure were done using ABAQUS/Explicit, v. 6.10. Cylinders were modeled using S4R shell elements (4-node, reduced integration, finite membrane strain) and consisted of extruded tubes with flat end caps. The simulation material model incorporated isotropic strain hardening, using the stress-strain relationship reported by Beese et al. [7].

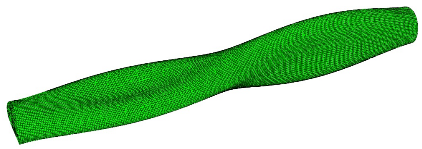
In order to study the effect of mode shape on energy dissipation, a relatively long, thin cylinder geometry was selected such that it naturally collapsed in mode 2. A static buckling analysis was performed on the perfectly circular cylinder to determine the first few buckling modes. For this particular example cylinder, the first five buckling mode shapes are shown in Fig. 7.

Modes 2A and 2B have the same 2-lobe cross-sectional shape as mode 2, but the cylinder length is subdivided into 2 or 3 segments rotated 90 degrees relative to each other. Similarly, mode 3A is identical to mode 3 but with the length subdivided into 2 segments. These intermediate modes exist for this cylinder because the length-to-radius ratio is relatively large ($\frac{L}{R} = 10.8$).

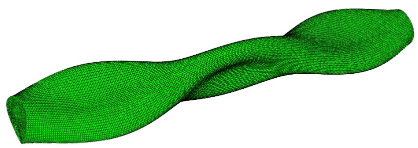
Following the static buckling analysis, a dynamic analysis was run with the minimum buckling pressure applied to the perfectly circular cylinder as a uniform hydrostatic load. As



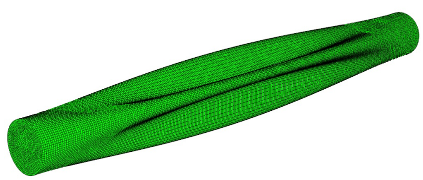
(a) Mode 2.



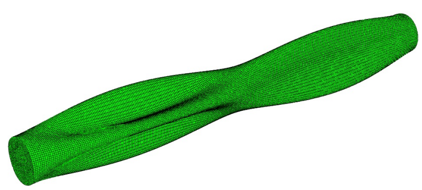
(b) Mode 2A.



(c) Mode 2B.



(d) Mode 3.



(e) Mode 3A.

Fig. 7: First five buckling modes for example cylinder.

expected, the cylinder collapsed in mode 2, and the plastic dissipation and external work were evaluated. The same cylinder was then subjected to initial geometric imperfections to trigger collapse in higher modes. The geometric imperfections were introduced as individual mode shapes, with maximum magnitude $\Delta = 1$ mm.[2] Dynamic simulations were run with mode 2A, 2B, and 3 imperfections, and resulting energy dissipation and external work are plotted in Fig. 8.

The external work, W, is nearly independent of collapse mode (within 4%), meaning that the final change in volume is about the same for all modes. However, the plastic energy dissipation, E_{int}, varies by up to 10% of the baseline value (i.e., the perfect cylinder with no imperfections). The implosion pulse energy can be roughly approximated as $W - E_{int}$ (neglecting the contribution of E_{air}). The ratio $\frac{E_{int}}{W}$ provides another measure of implosion pulse severity (higher $\frac{E_{int}}{W}$

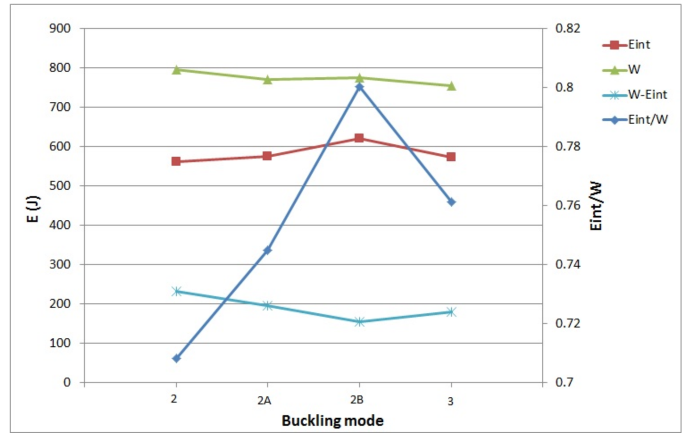


Fig. 8: Effect of collapse mode on plastic energy dissipation and external work ($\frac{\Delta}{R} = 0.08$).

corresponds to less severe pulse). In this example, mode 2B clearly has the minimum $W - E_{int}$ and maximum $\frac{E_{int}}{W}$, which indicates the minimum implosion pulse energy for this cylinder. It should be noted that Fig. 8 represents energies over the entire implosion event, so the results cannot be directly compared to the analytic solutions presented earlier, which only included phase 1.

The optimal mode yielding minimum pulse energy will vary with the cylinder geometry and material. Ideally, the designer should conduct an analysis similar to that described above to determine the mode of minimum pulse energy for the specific implodable of interest. However, Fig. 8 shows that some degree of pulse reduction will be realized for any mode higher than the natural mode 2. Thus, if a full analysis is not practical, the designer can choose any higher mode and be assured of some pulse reduction.

Next, the effect of the imperfection magnitude was investigated for the mode of minimum pulse energy (mode 2B in this case). Seven different values of $\frac{\Delta}{R}$ were evaluated, from 0 (perfect geometry) to 0.08, and results are plotted in Fig. 9.

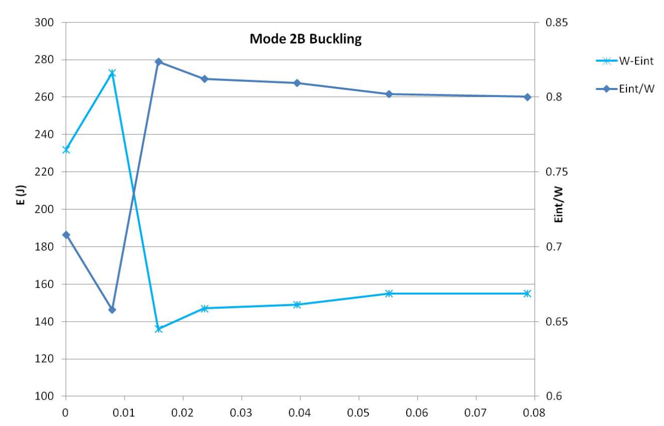


Fig. 9: Effect of imperfection magnitude on plastic energy dissipation and external work.

The case with $\frac{\Delta}{R} = 0.008$ actually generates a larger

[2]For this example, R=12.7 mm and $\frac{\Delta}{R} = 0.08$

implosion pulse (indicated by higher $W-E_{int}$ and lower $\frac{E_{int}}{W}$) than the baseline case with no imperfection. This is because the imperfection is not large enough to trigger the desired collapse mode 2B. The cylinder collapses in the natural mode 2 like the baseline case, but the imperfection serves to lower the plastic energy dissipation. The result is a larger implosion pulse. The minimum implosion pulse is seen at about $\frac{\Delta}{R} = 0.015$. The pulse energy is essentially constant above that magnitude of imperfection. For this example, the difference between the minimum pulse and the baseline with no imperfection is 77 J, or 33% of the baseline value.

Of course, intentionally causing geometric imperfections in the cylinder reduces the overall strength and resistance to buckling compared to the perfect geometry. In order to minimize the strength reduction, it is desirable to use the minimum imperfection magnitude necessary to achieve the desired mode shape. Figure 9 shows that there is a clear minimum value of $\frac{\Delta}{R}$ that forces the desired mode shape and also yields the smallest implosion pulse (in this case, $\frac{\Delta}{R} = 0.015$). For this example, applying the mode 2B imperfection with magnitude $\frac{\Delta}{R} = 0.015$ reduces the critical buckling pressure from 6.37 MPa to 6.34 MPa, a decrease of only 0.5%. This reduction is deemed acceptable, given the benefit of reducing the implosion pulse severity by 33%.

It may be difficult to manufacture precise mode shape imperfections into a real implodable. An alternative way to introduce imperfections is to create local stress concentrations (weak spots) in the structure in such a way as to trigger the desired collapse mode. For example, to trigger mode 3 collapse, the wall thickness of the center cross-section could be thinned slightly by grinding flat spots onto the outer cylinder surface at three points spaced 120 degrees apart (see Fig. 10). The flat spots in this figure are greatly exaggerated to be more visible. This method will have a greater detrimental effect on the overall collapse strength of the structure than the precise mode shape imperfection method described earlier. Therefore, the effect of the imperfections on the overall structural design must be carefully considered.

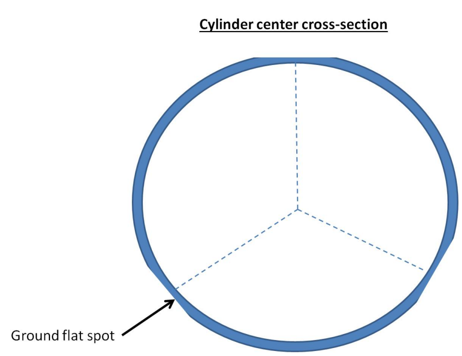


Fig. 10: Alternative method to manufacture imperfections to trigger mode 3 collapse.

Alternatively, a thin layer of energy absorbing material (such as a honeycomb structure) could be added to the inner surface of the cylinder. The energy absorbed by this layer would add to the plastic energy dissipation of the main cylinder structure and reduce the kinetic energy (and therefore the implosion pulse energy). Neither this nor the alternative defect method described have been quantitatively investigated, and are left as future work.

IV. Methods to Increase E_{air}

The compression of the internal gas (typically air) is approximated as quasistatic and adiabatic. The energy required for compression to a final volume V can then be calculated as:

$$E_{air} = \frac{p_{i0} V_0}{\gamma - 1} \left[\left(\frac{V_0}{V} \right)^{\gamma - 1} - 1 \right] \tag{5}$$

where p_{i_0} and V_0 are initial conditions of internal pressure and volume, and γ is the ratio of specific heat capacities for the gas ($\gamma = c_P/c_V = 1.4$ for air, where c_P is the specific heat capacity at constant pressure and c_V is the specific heat capacity at constant volume). Equation (5) provides insight into possible ways to increase E_{air}.

A. *Pressurize the Implodable Interior*

The compression energy varies directly with the initial internal gas pressure, p_{i_0}. Pressurizing the implodable interior above atmospheric pressure will increase the energy required to cause the same volume change. As the initial internal pressure of a given cylinder is increased, the buckling pressure increases by an equal amount, because buckling and collapse are caused by the differential pressure felt across the cylinder wall. Thus, increasing the internal pressure allows either (1) maintaining the cylinder geometry and operating at deeper depth, or (2) maintaining the same operating depth and reducing the wall thickness of the cylinder. These two cases are evaluated separately below.

Constant Cylinder Geometry

For this analysis the external hydrostatic pressure, p_{hyd}, is varied from the reference value by the same amount as the initial internal pressure, p_{i_0}, to maintain the same differential pressure across the cylinder wall. The reference condition is an initial internal pressure of 1 atmosphere (0.1 MPa), yielding an implosion pulse energy designated as E_0. The pulse energy was calculated over a range of internal pressures for two example cylinders. The two cylinders are the same material, but cylinder B is thinner relative to its radius. The non-dimensional pulse energy ($\frac{E}{E_0}$) is plotted vs. the non-dimensional pressure ($\frac{p_{i0}}{p_{hyd}}$) in Fig. 11. As expected, the pulse energy decreases as initial internal pressure increases. There exists a value for p_{i_0} above which the cylinder will not fully implode and no implosion pulse will occur.

Constant Collapse Depth

For this analysis, the internal pressure was increased and the cylinder wall thickness, h, was correspondingly decreased to maintain a constant initial collapse pressure (corresponding to operating depth). This is more realistic than the previous approach, because in most real applications, an implodable is designed for a specified operating depth. The ABAQUS numerical model was used to evaluate the buckling pressure of the two example cylinders over a range of wall thicknesses. The buckling pressure was subtracted from the constant hydrostatic pressure to give the required internal pressure corresponding

Cylinder A	Cylinder B
L=137.3 mm	L=183.4 mm
R=12.7 mm	R=19.07 mm
h=0.889 mm	h=0.714 mm
σ_0=351 MPa	σ_0=351 MPa

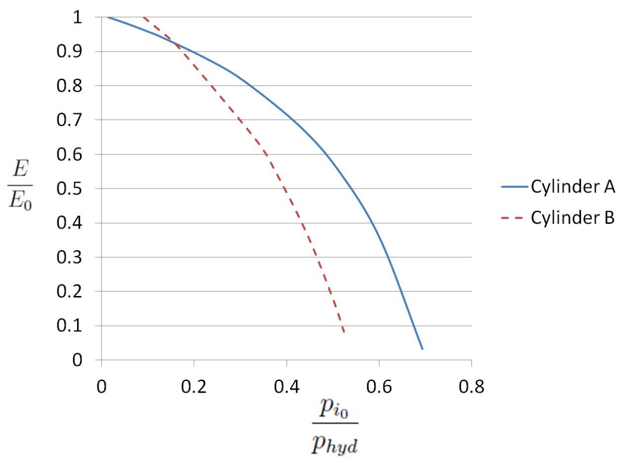


Fig. 11: Effect of varying internal pressure on implosion pulse energy (constant cylinder geometry).

Cylinder A	Cylinder B
L=137.3 mm	L=183.4 mm
R=12.7 mm	R=19.07 mm
h=0.889 mm	h=0.714 mm
σ_0=351 MPa	σ_0=351 MPa

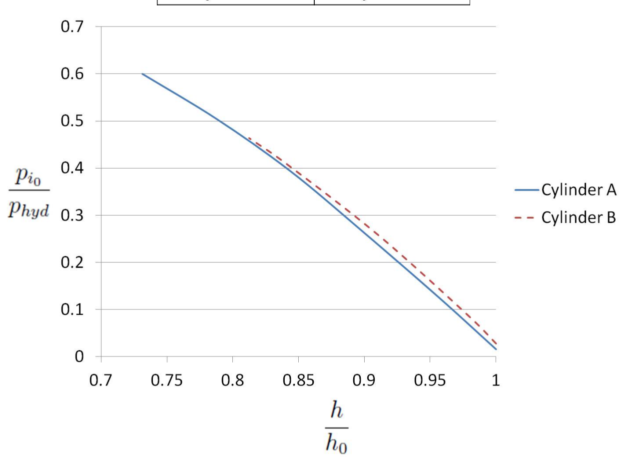
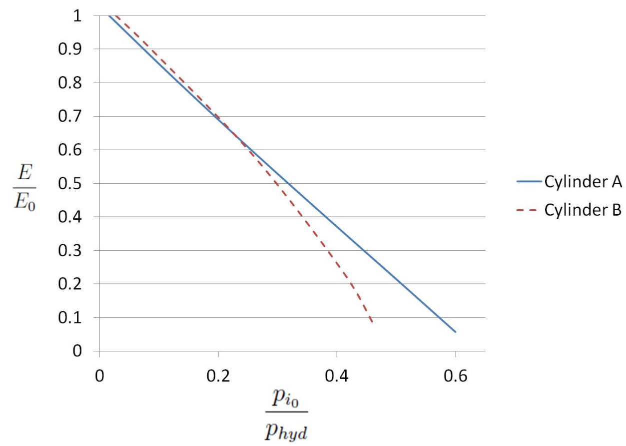
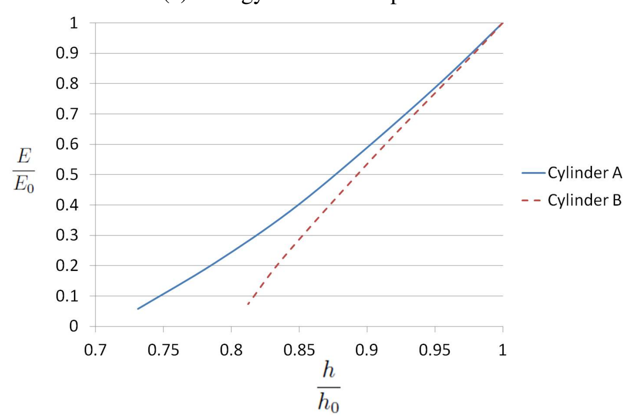


Fig. 12: Effect of varying internal pressure on cylinder wall thickness (constant collapse depth).

Cylinder A	Cylinder B
L=137.3 mm	L=183.4 mm
R=12.7 mm	R=19.07 mm
h=0.889 mm	h=0.714 mm
σ_0=351 MPa	σ_0=351 MPa



(a) Energy vs. internal pressure.



(b) Energy vs. thickness.

Fig. 13: Effect of varying internal pressure and thickness on implosion pulse energy (constant collapse depth).

to each thickness. The reference condition is the original wall thickness (h_0) with an internal pressure of 1 atmosphere (0.1 MPa). The non-dimensional pressure ($\frac{p_{i_0}}{p_{hyd}}$) is plotted vs. the non-dimensional thickness ($\frac{h}{h_0}$) in Fig. 12. The two cylinders show a nearly identical inverse relationship.

The pulse energy was then calculated over a range of wall thicknesses and corresponding internal pressures for the two example cylinders. The non-dimensional pulse energy ($\frac{E}{E_0}$) is plotted vs. the non-dimensional pressure ($\frac{p_{i_0}}{p_{hyd}}$) and non-dimensional thickness ($\frac{h}{h_0}$) in Fig. 13. The plots indicate that the pulse energy decreases as initial internal pressure increases and wall thickness decreases. There exists a value for p_{i_0} above which the cylinder will not fully implode and no implosion pulse will occur.

Limitations on Internal Pressure

The previous analysis shows that increasing the initial internal pressure, accompanied by either a corresponding increase in operating depth or a corresponding decrease in wall thickness, lowers the implosion pulse energy. If the internal pressure is high enough, complete implosion can be prevented. However, there are practical limits to the internal pressure. First, the contents of the implodable (electronics, batteries, sensors, propulsion systems, etc.) must be able to survive and function in the pressurized atmosphere. If the implodable is manned, then the human response to pressure, and the necessary subsequent decompression, must be considered. Second, internal pressure causes structural stresses opposite in sign from the stress caused by external pressure. In some cases, the stress from internal pressure could become design-limiting, particularly at shallow depths.

B. Vary Gas Composition

An alternative way to increase E_{air} is by changing the composition of the gas, which changes the ratio of specific heats (γ). From Eq. (5), it follows that a higher value of γ gives a higher value of E_{air}, for the same volume ratio. The noble gases (helium, neon, argon, krypton, xenon, and radon) all have $\gamma = 1.67$, compared to air with $\gamma = 1.4$. For the example cylinders analyzed in this work, the volume ratio at the end of phase 1 was approximately $\frac{V_0}{V} = 1.3$. For this volume ratio, increasing γ from 1.4 to 1.67 increases E_{air} by 3.7%. Therefore, using a noble gas instead of air as the internal medium would cause a slight increase in E_{air} and a corresponding decrease in implosion pulse energy.

V. Conclusions

The previous analysis shows that the severity of the implosion pulse can be significantly reduced by incorporating the following design techniques:

- Introduce geometric imperfections to trigger higher collapse modes
- Pressurize the implodable interior
- Substitute a noble gas for air in the interior

For the example cylinder considered, triggering a higher collapse mode reduced the implosion pulse by 33%. The benefit from pressurizing the interior is bounded by other limitations on internal pressure, such as equipment and human survivability. The higher the internal pressure, the greater the pulse reduction. Use of a noble gas yielded a slight pulse reduction.

The effectiveness of these design techniques will vary with each specific implodable's unique set of design specifications and constraints. However, these methods can be applied, in some degree, to any implodable to reduce the effect of implosion on adjacent structures or personnel.

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Appendix

The analytic solution for plastic energy dissipation (bending only) during phase 1 of mode 3 collapse was derived, using the assumptions and methodology developed by Gish and Wierzbicki [4]. The mode 3 cross-section is treated as six rigid segments connected by six plastic hinges, as shown in Fig. 6. Since the segments are rigid, all energy dissipation occurs at the plastic hinges. The instantaneous rate of bending energy dissipation for a single hinge is given by:

$$\dot{E}_b = 2M_0\dot{\alpha} \tag{6}$$

where $M_0 = \sigma_0 h^2/4$ is the fully plastic bending moment for a rigid, perfectly plastic material and α is the half-angle between two adjacent segments at the hinge. The cross-section contains six plastic hinges and the angles will be identical at all hinges, so the total rate of bending energy dissipation for a single cross-section is:

$$\dot{E}_b = 12M_0\dot{\alpha} \tag{7}$$

The bending energy dissipated from t=0 to t is found by integrating:

$$E_b = \int_0^t \dot{E}_b dt = 12M_0\left(\frac{\pi}{3} - \alpha\right) \tag{8}$$

where $\frac{\pi}{3}$ is the initial value of α. The total bending energy for the entire cylinder is found by integrating Eq. (8) over the cylinder length:

$$E_{b,total} = 2\int_0^L 12M_0\left(\frac{\pi}{3} - \alpha\right)dx \tag{9}$$

The angle α is a function of longitudinal position x, and is closely approximated by the linear function:

$$\alpha(x) = -1.009\left(\frac{w_0}{R}\right)\left(1 - \frac{x}{L}\right) + 0.9747 \tag{10}$$

Substituting Eq. (10) into Eq. (9) and evaluating the integral gives:

$$E_{b,total} = 12.108M_0L\left(\frac{w_0}{R}\right) \tag{11}$$

Phase 1 ends when $\frac{w_0}{R} = 1$. Substituting the expression for M_0 gives the final result for energy dissipation at the end of phase 1 for mode 3 collapse:

$$E_{b,total} = 3.027\sigma_0h^2L \tag{12}$$