Co-prime Comb Signals for Active Sonar

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***Abstract*—This paper presents an active sonar waveform that achieves range-Doppler performance similar to a uniform frequency comb, but uses far fewer tones to do so. The trade-off for this reduction in occupied bandwidth is a larger bandwidth extent. Co-prime comb signals consist of tones at non-uniformly spaced frequencies according to a 2-level nested co-prime array structure. Specialized non-matched filter processing enables recovery of an ambiguity surface similar to that of a uniform comb, but using fewer tonal components. This reduction in occupied bandwidth offers potential benefits such as sharing, interference avoidance, and Signal-to-Noise Ratio (SNR) improvements in both peak- and total-power-limited scenarios.**

I. Introduction

Reverberation is a major obstacle for active sonar systems, especially in shallow water. There are two conventional signal design approaches used to mitigate reverberation. The first involves using a wide-band pulse with a flat spectrum (e.g. linear FM) and pulse compression to spread the reverberation energy over the bandwidth of the pulse. The drawbacks to this approach are that these waveforms often have limited Doppler resolution, and also that in multipath delay-spread environments, echo compression gains may be limited [1]. The second approach involves exploiting the Doppler resolution of a long duration continuous wave (CW) pulse. Assuming a stationary platform, non-moving reverberation returns form a peak at zero-Doppler with a rapid falloff with increasing Doppler shift. Echoes from targets with sufficient velocities will appear in a region where the reverberation power is low. The drawback to this approach is that CW pulses have poor range resolution.

Frequency comb waveforms and waveforms with comb-like spectra enable achieving the reverberation mitigation benefits of both wide-band and long-duration CW pulse approaches [2]. Comb waveforms also offer good delay and Doppler resolution. The cost, however, at least in the case of a uniform comb, is the appearance of periodic range and velocity ambiguities in the delay-Doppler surface seen in the schematic ambiguity diagram shown in Fig. 1. Methods for spreading and suppressing these ambiguities have also been studied [1], [3]. Though these periodic ambiguities can be troublesome, they may be accommodated with careful parameter selection.

A majority of the previous work studying comb waveforms for active sonar involve using matched-filter-based processing techniques. Matched filtering is optimal for a single target in white noise and may work well in a host of scenarios, but it is not necessarily the best processor in multi-target scenarios in reverberation. In radar, "mis-matched" filter approaches have been used to improve ranging performance in multi-target scenarios [4]. Mis-matched filtering involves using a processor with filter weights that are not matched to the transmitted waveform.

In this paper we present the co-prime comb signal and specialized non-matched filter processing to achieve delay-Doppler performance of a uniformly spaced frequency comb, but using far fewer frequencies. Co-prime frequency sampling frees up bandwidth for other uses, such as spectrum sharing or interference avoidance, at the cost of requiring a larger bandwidth extent. Transmitting fewer frequencies also offers potential for SNR improvements in both peak- and total-power-limited scenarios. In addition, co-prime combs may be used in continuous active sonar (CAS) scenarios where waveforms are transmitted continuously at 100% duty cycle.

II. Frequency Comb Signals

The general expression for a comb waveform with K tonal components is given by

$$s_0(t) = \sum_{k=1}^{K} a_k e^{j(\omega_k t + \phi_k)} \tag{1}$$

where $\{\omega_k\}_{k=1,\ldots,K}$ represents each component frequency, $\{a_k\}_{k=1,\ldots,K}$ are design amplitudes and $\{\phi_k\}_{k=1,\ldots,K}$ are design phases. In conventional pulsed systems, temporal windowing is applied to the expression above to form short pulses and long delays are placed between transmissions to achieve a sufficient unambiguous range. Our work focuses on CAS implementations where the transmitter is sounding continuously. The benefits of CAS systems is that they offer higher revisit rates and enable using the full transmit power available over a given time interval.

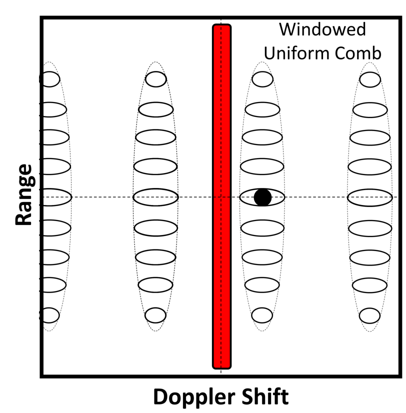

Fig. 1. Schematic range-Doppler response of a moving target with a windowed uniform comb waveform. The red strip indicates the non-moving reverberation.

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Different comb structures can be achieved by varying the parameters of each tonal component. In this paper we are concerned with adjusting number of tones K and their corresponding frequencies. The idea is to use frequency spacings based on a co-prime array to reduce the number of tones it takes to achieve equivalent performance to a uniform comb with constant frequency spacing.

A. *Uniform Comb Signals*

A uniform comb with K equally-spaced frequencies is achieved by setting

$$\omega_k = 2\pi(f_0 + (k-1)\Delta f) \quad (2)$$

where f_0 is the fundamental frequency and Δf is a design parameter. The parameter Δf effects the spacing of the range and Doppler ambiguities shown in Fig. 1. Fig. 2 (top) shows an example time series and Fig. 3 (bottom) shows an example spectrum for a uniform comb with $K = 36$, $f_0 = 0$ Hz, $\Delta f = 30$ Hz and $\{\phi_k\}_{k=1,\dots,K} = 0$. Note how the spectral periodicity of the waveform causes the time series to resemble an impulse train with a high peak-to-average power ratio. In practical scenarios, peak transmission power is limited which therefore limits the SNR achievable with an unmodified uniform comb. Previous work has shown that the peak-to-average power of comb waveforms may be reduced in a number of different ways such as reducing the number of tones, windowing, or adjusting tone frequencies or start phases [1], [2], [5]. Co-prime combs can reduce the peak-to-average power ratio because they require fewer tones than uniform comb signals to achieve similar performance.

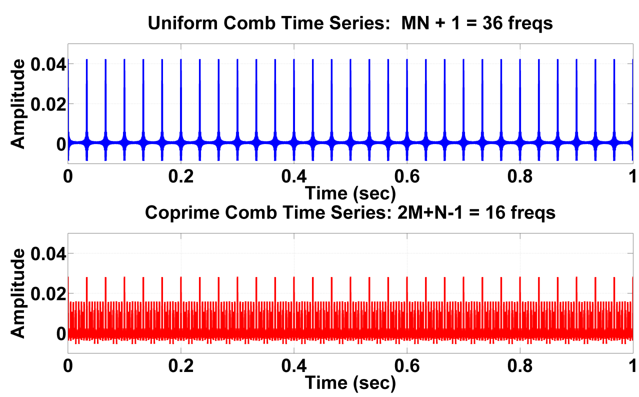

Fig. 2. Example time series for a uniform comb with 36 frequencies (top) and a co-prime comb with 16 frequencies (bottom).

B. *Co-prime Comb Signals*

A nested co-prime array is formed by taking the union of uniform arrays whose spacings are co-prime factors of one another. Co-prime comb signals are comb signals whose frequencies are separated based on a nested co-prime array structure. For simplicity, we only consider a 2-level nested co-prime array structure. Choosing two co-prime integers M and N, we form two subarrays with spacings Md and Nd with N and $2M$ elements respectively, where d is the nominal array spacing. In the case of spatial sampling, d is often chosen to be $\lambda/2$ to meet the spatial Nyquist criteria. In our case we are interested in sampling a bandwidth, so we let $d = \Delta f$. The number of elements in each subarray are carefully chosen to ensure a filled co-array of length $MN + 1$ as laid out by Pal and Vaidyanathan [6]. Note that the resulting array contains $2M+N-1$ elements but achieves a filled co-array of length $MN + 1$. A 2-level nested co-prime comb waveform is achieved by setting

$$\{\omega_k\}_{k=1,\dots,2M+N-1} = \{\omega_0 + Mn\Delta\omega, 0 \le n \le N-1\} \bigcup \{\omega_0 + Nm\Delta\omega, 1 \le m \le 2M-1\}. \quad (3)$$

Fulfilling the criteria necessary for a filled $MN+1$ length co-array is crucial for success of the specialized non-matched filter processing approach laid out in section III. This is what ensures that all of the frequency difference "lags" are present that are necessary to achieve an ambiguity function similar to a uniform comb with $MN+1$ frequencies, while utilizing only $2M + N - 1$ frequencies. The cost, however, is that co-prime combs require roughly two times the bandwidth extent of the equivalent uniform comb.

Fig. 3 shows example spectra of a co-prime comb with $M = 5$, $N = 7$, and $\Delta f = 30$ Hz and the comparable uniform comb. The green shaded regions highlight areas of bandwidth that have been restored by the co-prime structure which could potentially be used for spectrum sharing or interference avoidance. Also note that these regions could be shifted by moving the $M\Delta f$-spaced subarray in frequency. The orange shaded region highlights the increase in bandwidth extent required by the co-prime comb.

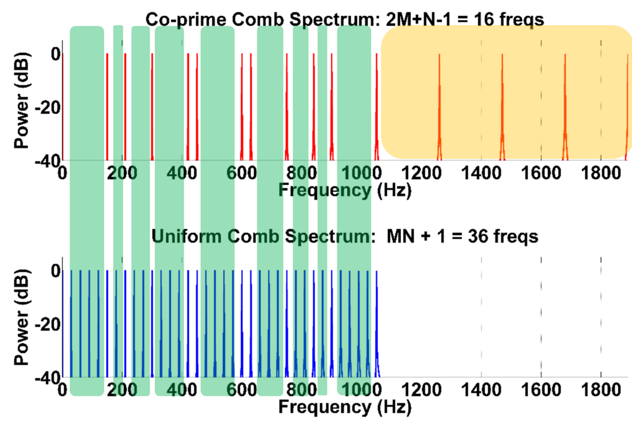

Fig. 3. Example spectra for uniform and co-prime comb signals that achieve similar range-Doppler performance. Green shading indicates recovered bandwidth and the orange shading indicates the increased bandwidth extent required by the co-prime comb.

III. Co-prime Comb Processing Algorithm

Historically, comb waveforms have been processed using matched filtering approaches [1], [3], [7], [8]. The matched filter is optimal when detecting a single target in white noise, but is not necessarily the best processor when there are multiple targets and correlated reverberation returns. The idea behind co-prime combs and our processing technique involves transmitting an aperiodic co-prime signal, receiving it, and then processing it in a way that yields an output resembling that of a periodic uniform comb signal. This is done by transforming the ranging problem into a dual of the direction of arrival (DOA) estimation problem with a spatial co-prime array. From

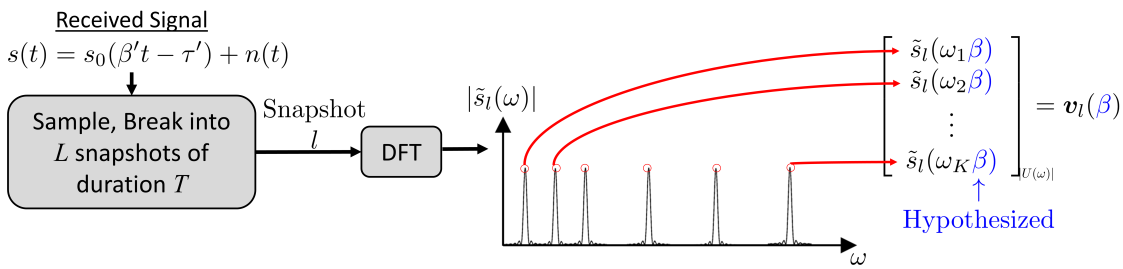

Fig. 4. Block diagram of snapshot formation process for co-prime comb signal processing.

there, a co-prime array processing technique is used to achieve performance similar to that of a uniform comb.

Neglecting scattering amplitude and fading, the general expression of a single point target return with delay τ' (modulo the maximum unambiguous delay) and radial velocity v' in noise is given by

$$s(t, \beta', \tau') = \sum_{k=1}^{K} a_k e^{j[\omega_k(\beta' t - \tau') + \phi_k]} + n(t) \tag{4}$$

where the Doppler factor $\beta' = 1 - 2v'/c$.

From here, the received signal is processed using the steps below. Note that steps 1-3 are also shown in the block diagram in Fig. 4

1) Sample the received signal at F_s and divide it into L snapshots of duration T, with the lth snapshot starting at time t_l.
2) Hypothesize β and take a DFT of each of the snapshots.
3) Retrieve the frequency bins located at $\{\omega_k\beta\}_{k=1,\ldots,K}$. Stack each of the resulting sets of K DFT coefficients into a vector $\boldsymbol{v}_l(\beta)$.
4) For each snapshot, remove the effects of delay and design phase to form

$$\boldsymbol{u}_l(\beta) = \text{diag}([e^{j(\omega_1\beta t_l - \phi_1)}, \ldots, e^{j(\omega_K\beta t_l - \phi_K)}]^T)\boldsymbol{v}_l(\beta).$$

Now form the sample covariance matrix

$$\boldsymbol{R}_{\boldsymbol{u}}(\beta) = \frac{1}{L}\sum_{l=1}^{L} \boldsymbol{u}_l(\beta)\boldsymbol{u}_l^H(\beta). \tag{5}$$

where $(\cdot)^T$ indicates a transpose and $(\cdot)^H$ indicates a Hermitian transpose. Note that $\boldsymbol{R}_{\boldsymbol{u}}(\beta)$ is $K \times K$ and in the co-prime case $K = 2M + N - 1$, representing each tone. Now the goal is to use this $(2M + N - 1) \times (2M + N - 1)$ matrix to estimate the $(MN + 1) \times (MN + 1)$ covariance matrix that would be obtained when using a uniform comb with $MN + 1$ tones. This is achieved by using a modified version of rank-enhanced spatial smoothing that exploits the hole-free co-array property [6]. We call our implementation of this technique "rank-enhanced spectral smoothing" which uses the steps explained below:

1) Determine how frequency differences (lags) map to $\boldsymbol{z}_{K^2\times 1}(\beta) = \text{vec}(\hat{\boldsymbol{R}}_u(\beta))$.
2) Average the frequency "lag" redundancies.
3) Re-order frequency lags from $-MN$ to $+MN$.
4) Separate the resulting vector into $MN + 1$ different subarrays $\{\boldsymbol{z}_{i,1}(\beta)\}_{i=1,\ldots,MN+1}$ of length $MN + 1$.
5) Calculate $\boldsymbol{R}_{ss}(\beta) = \frac{1}{MN+1}\sum_{i=1}^{MN+1} \boldsymbol{z}_{1,i}(\beta)\boldsymbol{z}_{1,i}^H(\beta)$.

The resulting $(MN+1)\times(MN+1)$ matrix $\boldsymbol{R}_{ss}(\beta)$ is a full-rank positive definite estimate of the covariance matrix due to a uniform comb with $MN + 1$ frequencies. After forming $\boldsymbol{R}_{ss}$, the delay-Doppler response is determined in a manner similar to how a spatial array is steered using a steering vector. Instead of hypothesizing a particular angle of arrival, a delay τ is hypothesized and the equivalent $MN + 1$ length "steering" vector becomes

$$\boldsymbol{w}_{ss}(\tau) = [e^{j\omega_1\tau},\ e^{j\omega_2\tau}, \ldots,\ e^{j\omega_{MN+1}\tau}]^T. \tag{6}$$

The delay-Doppler response without rank-enhanced spectral smoothing can also be computed by using the $2M + N - 1$ length "steering" vector

$$\boldsymbol{w}_{\boldsymbol{u}}(\tau) = [e^{j\omega_1\tau},\ e^{j\omega_2\tau}, \ldots,\ e^{j\omega_{2M+N-1}\tau}]^T \tag{7}$$

together with $\boldsymbol{R}_{\boldsymbol{u}}(\beta)$. The conventional delay-Doppler response can now be determined for each hypothesized delay and Doppler using

$$\hat{P}_c(\beta, \tau) = \boldsymbol{w}^H(\tau)\hat{\boldsymbol{R}}(\beta)\boldsymbol{w}(\tau). \tag{8}$$

Other covariance-matrix-based processors could also be used in the place of conventional processing, such as minimum variance distortionless response (MVDR) or multiple signal classification (MUSIC). $\boldsymbol{R}_{ss}(\beta)$ is guaranteed to be full rank and positive definite due to how rank-enhanced spectral smoothing is performed.

It can be shown that the delay-Doppler surface described by (8) results in range ambiguities separated by

$$r_{amb} = c/2\Delta f \tag{9}$$

for conventionally-processed uniform combs and for co-prime combs processed using rank-enhanced spectral smoothing. In CAS scenarios, range ambiguities can be move outward by appropriately choosing Δf, as long as the tones remain resolvable. In pulsed scenarios, it may be desirable to choose a large Δf to concentrate range ambiguities into a small range extent and then use delays between transmissions to achieve a sufficient unambiguous range. The drawback of this approach is that it may cause the "achievable" range resolution to no longer be dependent on the bandwidth extent of the signal.

It can be shown that the delay-Doppler surface described by (8) results in velocity ambiguities separated by

$$v_{amb} = \frac{c\Delta f}{2f_c} \tag{10}$$

where $f_c = f_0 + MN\Delta f/2$. Wide spacings between Doppler ambiguities are desirable to achieve deep valleys in the reverberation spectrum away from the zero-Doppler ridge. By adjusting f_0 and Δf, Doppler ambiguities may be separated as much as the hardware will allow, while maintating a sufficient velocity resolution. However, range and velocity ambiguity spacings are coupled via the Δf parameter and this must be managed.

Co-prime comb signals can offer improved SNR over uniform combs in both peak- and total-power limited scenarios, while maintaining range-Doppler performance. As shown in the example in Fig. 2, transmitting fewer tones can reduce the peak-to-average power ratio therefore increasing SNR in peak-power-limited scenarios. In total-power-limited-scenarios, using fewer tones leaves more transmit power available per tone. The predicted asymptotic SNR improvement due to using a co-prime comb rather than the equivalent uniform comb is $20\log_{10}(\frac{MN+1}{2M+N-1})$ dB, which depends on the ratio of the number of tones in each waveform.

IV. Simulation Results

Simulations were performed to confirm that a co-prime comb with $2M+N-1$ frequencies can achieve range-Doppler performance similar to that of a uniform comb with $MN+1$ frequencies and are presented in subsection IV-A. Simulations were also performed to show SNR improvements in a total-power-limited scenario and are presented in subsection IV-B. Unless noted otherwise, all of the following simulations used $M = 5$, $N = 7$, $f_0 = 300$ Hz, $\Delta f = 2$ Hz, $T = 5$ sec, $L = 50$, $r' = 0$ km, and $v' = 0$ m/s with $SNR_{input} = 20$ dB with additive white Gaussian noise (AWGN). The co-prime comb used $2M+N-1 = 16$ tones distributed within a 126 Hz bandwidth extent, while the uniform comb used $MN+1 = 36$ tones distributed within a 70 Hz bandwidth extent.

A. Range-Doppler Responses

Fig. 5 shows the range-Doppler response of the matched-filter-processed uniform comb and Fig. 6 shows the range-Doppler response of the co-prime comb generated using rank-enhanced spectral smoothing. Visual comparison of the two surfaces reveals their similar range-Doppler performance. Range ambiguities appear in both plots across $v = 0$ that match the predicted separations of 0.375 km. The first set of velocity ambiguities are also present (smeared vertically across range) at the predicted separation of 4.478 m/s.

Zero-velocity slices (range profiles) across these two surfaces are plotted in Fig. 7 in blue and red, respectively. The results match so well that it is difficult to see the non-overlapping parts of the lines. The black line in this plot represents the matched filter output of same co-prime comb without rank-enhanced spectral smoothing. The drastic range side lobes of the match-filtered co-prime result are due to the non-uniform spacing of the tones and helps to clarify the need for specialized non-matched filter processing.

Fig. 8 shows slices across zero range for Fig. 5 and Fig. 6. Again note the good match between the uniform and augmented co-prime combs, especially at low velocities from ≈ 0 to 3 m/s. Side lobe levels at higher velocities (> 3 m/s) as well as the level of the first velocity ambiguity are lower for the augmented co-prime result. In a scenario with reverberation and a moving target, the co-prime comb's lower side lobes at these higher velocities could make it possible to detect targets that would be masked by reverberation returns with a uniform comb.

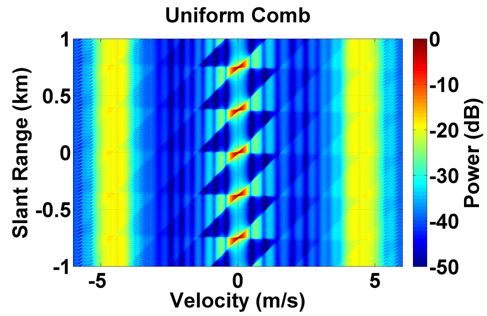

Fig. 5. Range-velocity surface for a matched-filtered, 36-tone uniform comb.

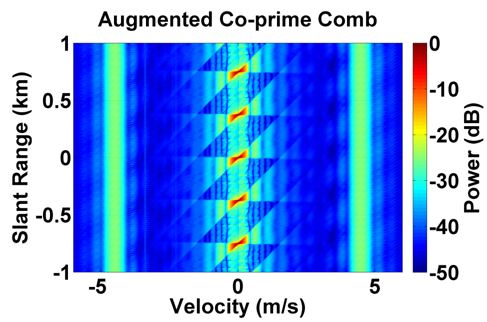

Fig. 6. Range-velocity surface for a 16-tone co-prime comb processed with rank-enhanced spectral smoothing.

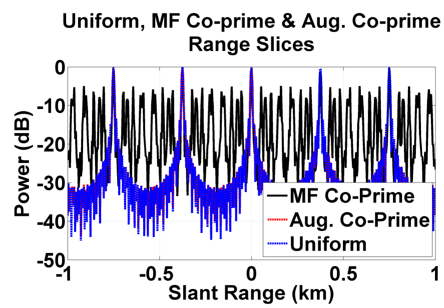

Fig. 7. Target range slices (range profiles) from Fig. 5 (red) and Fig. 6 (blue) as well as the response due to matched filtering the co-prime comb (black).

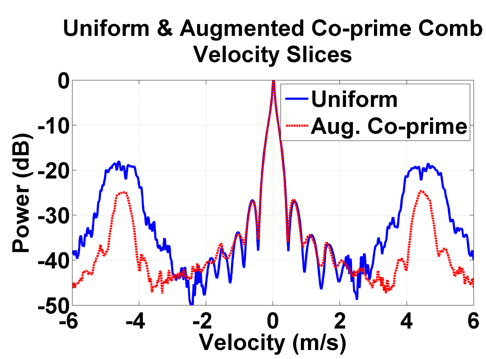

Fig. 8. Target velocity slices from Fig. 5 (red) and Fig. 6 (blue).

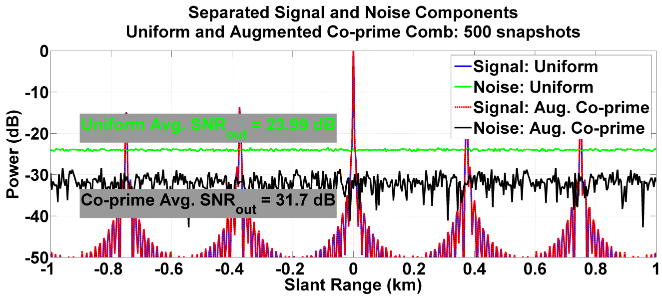

Fig. 9. Separated signal and noise processor outputs for both a uniform comb with matched filter processing (blue and green, respectively) as well as a co-prime comb processed using rank-enhanced spectral smoothing (red and black, respectively), $L = 500$, $SNR_{input} = -20$ dB.

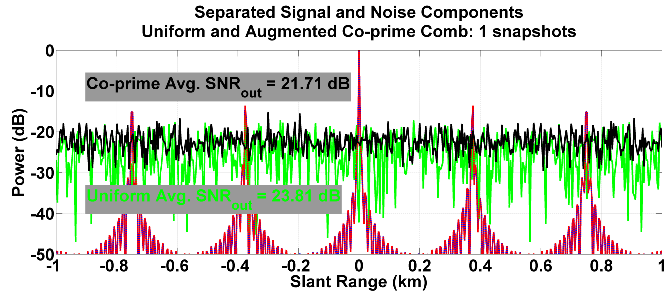

Fig. 10. Same curves as shown in Fig. 9 but using only a single snapshot, $L = 1$, $SNR_{input} = -20$ dB.

B. Signal-to-Noise Ratio Improvements

Fig. 9 shows separated signal and noise processor output components for both the co-prime comb using co-prime processing (dashed red and solid black, respectively) and the match filtered uniform comb (solid blue and solid green, respectively) using $L = 500$ snapshots. The total power of both waveforms were chosen to be the same to compare their performance in a scenario where the total power is limited. The large number of snapshots was chosen to approximate the asymptotic ($L \rightarrow \infty$) noise suppression performance. Comparing the signal components (dashed red and solid black) for both waveforms, we notice a good match which confirms that any noise suppression improvements do not come at the cost of sacrificing ranging performance. The gap between the noise outputs due to the uniform comb (green) and the noise output due to co-prime comb and processing (red) represents the asymptotic SNR improvement due to using the co-prime comb. Subtracting the average noise output levels yields $31.7 - 23.99 = 7.71$ dB which closely matches the predicted $20\log_{10}(\frac{MN+1}{2M+N-1}) = 7.04$ dB improvement.

Fig. 10 shows the same set of processor outputs shown in Fig. 9, but using a single snapshot. Again there is a good match between the uniform and co-prime comb signal components, but the noise levels have changed. The average SNR gains present in the asymptotic case are lost. Also notice the increase in the variance of the noise output level for the uniform comb. These effects are due a lack of snapshot support required to obtain a good spectral covariance estimate. Despite the increase in the noise output variance for the uniform comb, the peak noise output levels of the uniform and co-prime combs appear at similar levels indicating that co-prime combs suppress peak noise levels at least as well as a uniform comb.

V. Conclusion

A co-prime frequency comb waveform with accompanying specialized non-matched filter processing has been presented that offers range-Doppler performance similar to a uniform comb, but uses far fewer frequencies. Using fewer frequencies offers opportunities for spectrum sharing and interference avoidance, potential reductions in peak-to-average power ratio, as well as increased transmit power per tone leading to SNR improvements. Because range processing is dependent on tone to tone correlations, pulsed operation is not required, making this a promising waveform for CAS systems.

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