Active Sonar Target Tracking for Anti-Submarine Warfare Applications

J. Wang, A. von Trojan and S. Lourey
Defence Science and Technology Organisation
West Avenue
Edinburgh, SA 5111 Australia

***Abstract*-Tracking potential submarine targets using active sonar is a challenging task, due to the high level of clutter that results in a large number of false alarms. This can potentially overwhelm or critically limit the capabilities of an active sonar tracker.**

The focus of this paper is to investigate the performance of conventional tracking algorithms for both a single target and multiple targets in cluttered environments. The tracking algorithms considered in this study are the Extended Kalman Filter (EKF) and the Converted Measurements Kalman Filter (CMKF).

For single target in clutter the Probabilistic Data Association (PDA) is used, while for multiple targets in clutter, the Global Nearest Neighbour (GNN) data association algorithm is implemented.

A new previously unpublished version of the PDA is developed for use with the CMKF tracker to take into account that the measurement covariance matrix, innovation covariance matrix and filter gain are all dependant on the individual measurements in the gate at a given time, making the updated state estimate more complicated than that used for the EKF.

The performance of the tracking and data association algorithms is tested using Monte Carlo simulations.

I. Introduction

As modern submarines become quieter through the application of new technologies, such as hull coatings and better designed machinery mounts and propellers, it is increasingly difficult to rely on passive sonar to provide tracking information in both the wide area and tactical surveillance phases of an Anti-Submarine Warfare (ASW) mission. To counter this trend, active sonar, in particular, low to mid frequency active sonar, which offers the potential to provide a mid to long range detection capability, has been used in ASW operations. Examples of operational active sonar systems fitted on surface combatants include hull mounted arrays, towed active arrays and active sonobuoys.

Tracking potential submarine targets using active sonar is a challenging task, in particular, in the shallow, littoral environment, due to high levels of clutter resulting in a large number of false alarms. This can potentially overwhelm or critically limit the capabilities of an active sonar tracker.

An investigation of the performance of active sonar tracking systems against a submerged submarine in an acoustically complex shallow water environment may be useful in finding effective ways of establishing and maintaining tracks.

The focus of the paper is on the performance of conventional tracking algorithms for both a single target and multiple targets with clutter under different scenarios. Among the available tracking algorithms, the EKF and the CMKF are the most widely used. In the CMKF, the raw measurements in polar coordinates are converted to Cartesian coordinates. The conversion is statistically unbiased and linear so that the Kalman Filter can still be applied. Probabilistic Data Association (PDA) is used by many tracking systems to deal with false alarms. However, in the CMKF, the measurement covariance matrix, innovation covariance matrix and filter gain are not constant for all measurements at a give time as in the case of EKF, but dependant on the individual measurements in the gate. The usual updated state estimate and the updated state error covariance matrix equations in the filter no longer hold. To overcome this, a new previously unpublished CMKF-PDA algorithm is developed by deriving the expressions of the updated state estimate and the updated state error covariance matrix under the assumption of a single target with clutter. The GNN data association algorithm is adopted for multiple targets with cluttered scenarios.

This study is based on simulated data. Testing and analysis of the tracking and data association algorithms is undertaken and results are presented for various realistic simulated scenarios. A number of operationally meaningful metrics are defined to measure the performance of the tracking algorithms. Techniques for automatic track initiation and track termination are also implemented.

II. Tracking

A. Tracking Algorithms

The Kalman filter is the optimum estimator for the sequence of states produced by the model

$$\begin{aligned} X(k+1) &= F(k)X(k) + u(k) \\ z(k) &= H(k)X(k) + v(k) \end{aligned} \tag{1}$$

where $X(k)$ is the state vector of the model at time instant k, $F(k)$ is the known state transition matrix, $z(k)$ is the measurement vector, $u(k)$ is a random Gaussian noise vector with zero mean and covariance matrix $Q(k)$, and $v(k)$ is a random Gaussian noise vector with zero mean and covariance matrix $R(k)$. It is assumed that $u(k)$ and $u(m)$

978-1-4244-5222-4/10/$26.00 ©2010 CROWN

$(k \neq m)$ are independent. Likewise, $v(k)$ and $v(m)$ $(k \neq m)$, and $u(k)$ and $v(m)$ are independent. For many applications, the measurement equation in (1) is nonlinear

$$z(k) = f(X(k)) + v(k)\,. \qquad (2)$$

The EKF approach approximates (2) by taking the linear terms in the Taylor series expansion of the function at the predicted state estimate. For active sonar tracking, the system and measurement equations are

$$\begin{aligned} X(k+1) &= F(k)X(k) + u(k) \\ z(k) &= f(X(k)) + v(k) \end{aligned} \qquad (3)$$

where $X(k) = [x(k), y(k), \dot{x}(k), \dot{y}(k)]^T$ is the state vector and

$$F(k) = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

represents constant velocity with random perturbations and T is the time duration between consecutive sonar pings, and $z(k) = [range, bearing]^T$ with

$$f([x,y]^T) = \begin{bmatrix} \left[(x-x_R)^2 + (y-y_R)^2\right]^{1/2} \\ \arctan\left(\dfrac{x-x_R}{y-y_R}\right) \end{bmatrix}$$

where $[x_R, y_R]^T$ is the sonar location at the time.

Another approach to deal with the nonlinearity of the measurement equation is to use the CMKF method [1, 2, 3], which is used to overcome the bias introduced by the classical polar-to-Cartesian conversion. Let r_m and θ_m be the range and bearing measurements and

$$\begin{aligned} r_m &= r + v_r \\ \theta_m &= \theta + v_\theta \end{aligned}$$

where r and θ are the true range and bearing at the time and v_r and v_θ are independent Gaussian random noise variables with variance σ_r^2 and σ_θ^2 respectively. Let

$\lambda_\theta = E(\cos v_\theta) = \exp(-\sigma_\theta^2/2)$ and

$$\lambda'_\theta = E(\cos 2v_\theta) = \exp(-2\sigma_\theta^2).$$

The unbiased converted measurement vector is

$$X_m = \begin{bmatrix} \lambda_\theta^{-1} r_m \cos\theta_m \\ \lambda_\theta^{-1} r_m \sin\theta_m \end{bmatrix} \qquad (4)$$

The measurement equation for the active sonar CMKF is

$$X_m(k) = H(k)X(k) + w(k)$$

where

$$H(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \lambda_\theta^{-1}.$$

The measurement covariance matrix $R(k)$ takes the form [2] of

$$R(k) = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix},$$

where

$$\begin{aligned} R_{11} &= -\lambda_\theta^2 r_m^2 \cos^2\theta_m + \frac{1}{2}(r_m^2 + \sigma_r^2)(1 + \lambda'_\theta \cos 2\theta_m), \\ R_{22} &= -\lambda_\theta^2 r_m^2 \sin^2\theta_m + \frac{1}{2}(r_m^2 + \sigma_r^2)(1 - \lambda'_\theta \cos 2\theta_m), \\ R_{12} &= -\lambda_\theta^2 r_m^2 \sin\theta_m \cos\theta_m + \frac{1}{2}(r_m^2 + \sigma_r^2)\lambda'_\theta \sin 2\theta_m, \\ R_{21} &= R_{12.} \end{aligned} \qquad (5)$$

B. *Data Association Algorithms*

In the presence of false alarms and clutter, there is uncertainty as to which measurements are real and associated with a track. Data association algorithms attempt to resolve these ambiguities. A gate G around the filter predicted measurement is formed with a desired confidence level that the true measurement will fall into it. The gate is defined by

$$\nu(k+1)^T S^{-1}(k+1)\nu(k+1) \leq \gamma, \qquad (6)$$

where $\nu(k+1)$ is the innovation and $S(k+1)$ is the innovation covariance matrix. The value of the parameter γ can be selected from tables of the chi-square distribution [4], which also give the probability that the true measurement falls into the gate for the selected γ value. For active sonar tracking applications with range and bearing measurements, the number of degrees of freedom is 2. The choice of $\gamma = 16$ that we use, results in a probability of 0.9997 that the true measurement falls into the gate.

In the Nearest Neighbour (NN) data association algorithm, the measurement in the gate G closest to the predicted measurement is used to update the tracker. In the PDA, a weighted sum of all measurements in G is used to update the tracker. The weights are the calculated probabilities that the measurements originated from a target, based on statistical assumptions such as that the distribution of measurements in space is uniform and the distribution for the number of measurements in the gate is Poisson.

In most active sonar applications, multiple targets need to be dealt with because of the presence of various man-made and biological objects in the operational area. There have been several approaches to data association for multiple targets in clutter, such as Joint Probabilistic Data Association (JPDA) [4], Monte-Carlo Data Association (MCDA) [5] and Global Nearest Neighbour Data Association (GNN) [6]. The GNN is a generalisation of the NN algorithm, which can be used for multiple target scenarios. It is used in this paper because of its simplicity. The modified Munkres algorithm is used to assign measurements to tracks [7].

As shown in (5), the measurement covariance matrix $R(k)$ for the CMKF is dependent on each measurement. Thus, the innovation covariance matrix and filter gain are also dependent on the individual measurement for a given time. This does not allow the application of CMKF with PDA directly as in the case of EKF and PDA. A modified CMKF-PDA algorithm is developed to resolve this by deriving the expressions of the state estimate and the state error covariance matrix under the assumption of a single target with clutter. The modified equations are shown below

$$\hat{X}(k \mid k) = \hat{X}(k \mid k-1) + \sum_{i=1}^{m_k} \beta_i(k) W_i(k) \nu_i(k),$$

$$\begin{aligned} P(k \mid k) &= \beta_0(k) P(k \mid k-1) \\ &+ \sum_{i=1}^{m_k} \beta_i(k)[I - W_i(k) H(k)] P(k \mid k-1) \\ &+ \sum_{i=1}^{m_k} \beta_i(k) W_i(k) \nu_i(k) \nu_i(k)^T W_i(k)^T \\ &- \sum_{i=1}^{m_k} \beta_i(k) W_i(k) \nu_i(k) \sum_{i=1}^{m_k} \beta_i(k) \nu_i(k)^T W_i(k)^T \end{aligned}$$

A detailed derivation is given in Appendix A. Also, a slightly modified GNN algorithm is used for the CMKF tracker, to allow for the filter innovation covariance being dependant upon individual measurements in the gate for multiple targets scenarios.

C. Automatic Track Initialisation and Track Termination

Algorithms for automatic track initialisation, track mergence/maintenance and track termination are implemented. Over the surveillance region, a grid is drawn with a predetermined cell size. Initially, no tracks exist. A track is initiated, if for five consecutive pings, a detection is reported three or more times in a cell. The track initiation uses the two-point differencing with information from the two most recent detections. A track is terminated if there are three or more pings out of five consecutive pings for which no detections fall into the gate defined by (6).

III. Scenarios and Metrics

A. Simulation Scenarios

The simulated data used to test the tracking algorithms is generated from a scenario model [8], which may include multiple targets along with an active monostatic sonar. All of the entities, i.e. sonar and targets may be moving or stationary. The sonar pings at regular time intervals.

The target detection probability can be a function of range, depth and bearing. For each ping, whether or not the sonar produces target detections is determined by a draw from a uniform random distribution compared with the detection probability for each target. If the target is detected, then multiple returns to simulate scattering from the target may be generated, with the number of returns determined by drawing from a Poisson distribution. Gaussian noise is added to represent measurement error. If the target is not detected, the predicted state is used as the updated state in the tracker.

The simulated data also includes false i.e., non-target, detections. The number of false detections occurring at each ping is drawn from a Poisson distribution with the expected value depending on the probability P_{fa} of a false detection per range–bearing cell per ping set by the user. It is also possible to include persistent clutter, retaining a specified percentage of randomly selected false detections from the previous ping of the sonar. New false detections are distributed uniformly in range and bearing across the field of view of the sonar.

B. Tracking Performance Metrics

The following metrics are used to measure the performance of the tracking algorithms for different scenarios.

1. *The root mean square error*

   $$\mathbf{rms} = \sqrt{\frac{1}{n}\sum_{1}^{n}(x\mathbf{target} - x\mathbf{track})^2}$$

   is used as a measure of track accuracy, where *x*target is the true target location and *x*track is the track location, and *n* is the number of pings that the target is tracked. The *rms* error is also used to determine whether tracks are associated with a target. The threshold for the *rms* error is set so that any tracks whose *rms* error is greater than the threshold are set as false targets.
2. *Number of false tracks NfalseTracks.*
3. *Track Fragmentation* is the number of tracks associated with the target. When this is above 1, it means that the tracker is unable to output a single continuous track for the entire target trajectory [9].
4. *Tracker Latency* is the maximum time lag in seconds until the tracker starts tracking the target.
5. *Track Probability of Detection Pd* is defined as

$$\textbf{TPd} = \frac{\sum_{i=1}^{Ntt} Ti}{T}$$

where *Ntt* is the number of true tracks, and *Ti* is the time duration of the *i*-th true track, and *T* is the simulation time. *TPd* is the ratio of the total duration of all tracks associated with a target and the total simulation time. For each associated track, the time duration is defined as the difference in time of the last and first contact that the track associates. Time overlaps in tracks for the same target are removed [9].

IV. Simulation Results

To investigate the performance of the EKF and CMKF trackers for both single target and multiple targets applications, a Monte Carlo simulation is undertaken. Three scenarios are considered. In all scenarios, a hull-mounted sonar is fitted on a ship (the own ship), which emits Linear Frequency Modulated (LFM) pings. Each scenario consists of 46 pings at 40 second intervals. Thus the total simulation time is 1800 seconds. The hull-mounted sonar has a mean range error of 40 m and a mean bearing error of 5 degrees. The own ship travels due north at a speed of 12 knots or approximately 6.17 m/s. The three scenarios are described below and illustrated in Figure 1.

Scenario One

In Scenario One, a single subsurface target moves in a nominally straight line with a start location of [5000, 16000] m and velocity [-2,-1.5] m/s

Scenario Two

In Scenario Two, a single manoeuvring surface target travels in two nominally straight legs, starting from [-1500, 14500] m with a velocity of [5, 3.33] m/s for 400 seconds, then changes direction and travels with a velocity of [-3.6, 1.8] m/s

Scenario Three

Scenario Three consists of the targets in Scenario One and Scenario Two, plus an additional stationary target located at [-1000, 11000] m/s.

The probability of detection is fixed at Pd=0.7 and the Pfa=2.0e-05. The number of multiple returns for each target is set to 2. The percentage of clutter which persists to the next ping is set to 30 %.

A tool has been developed in Matlab, which automates the entire tracking process using the algorithms described in Section II. For Scenarios One and Two, only one target appears in the area of interest. The EKF-PDA and CMKF-PDA are tested for these scenarios. Figures 2-5 show typical results of these trackers for Scenarios One and Two. In these plots, the own ship trajectory is shown in BLUE, the true target trajectories are in GREEN, whilst the estimated target trajectories, i.e. the tracker outputs, are in RED.

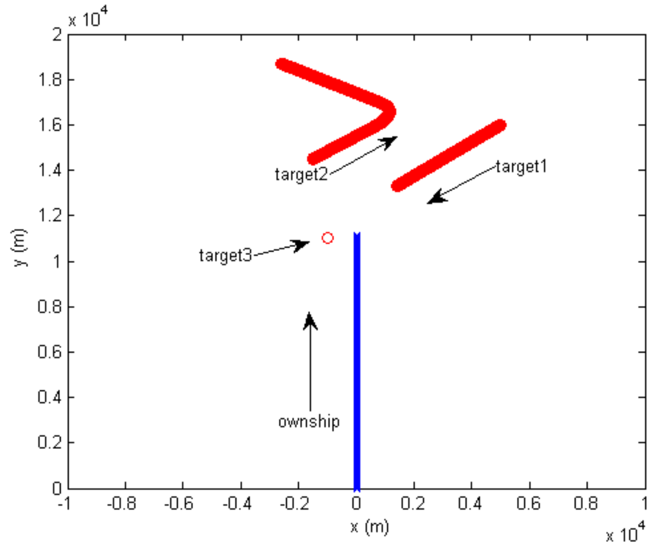

Figure 1. Scenarios consisting of ownship with onboard sonar and choice of three targets.

Figures 2-5 show that the trackers have some track latency for both the straight line and manouevering targets in initiating the target track. The starting time difference between the true target track (GREEN) and the estimated target track (RED) shows the track latency. The trackers also have problems in maintaining the track once the target has made its manouever. Several false tracks are generated for these scenarios. The CMKF tracker maintains the target track a little better than the EKF, but overall their performance is very similar.

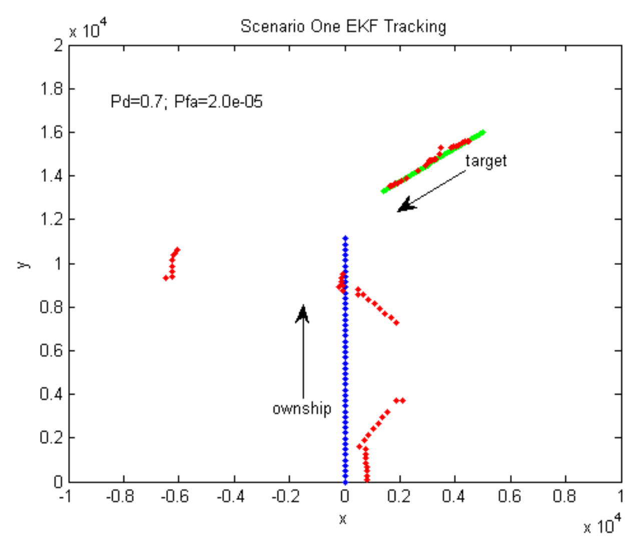

Figure 2. Tracks produced using the EKF-PDA tracker for Scenario One.

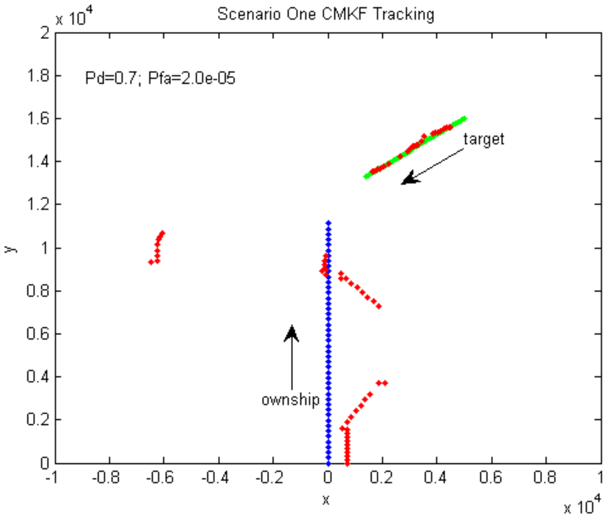

Figure 3. Tracks produced using the CMKF-PDA tracker for Scenario One.

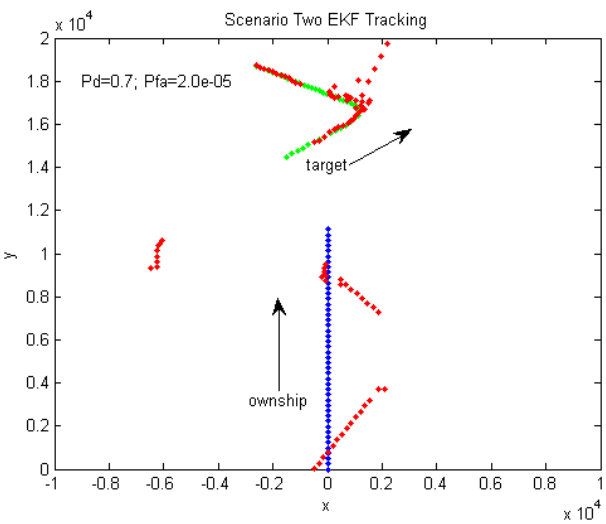

Figure 4. Tracks produced using the EKF-PDA tracker for Scenario Two.

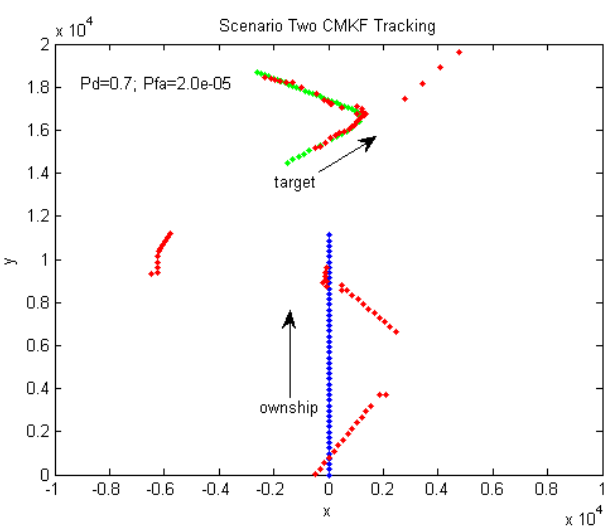

Figure 5. Tracks produced using the CMKF-PDA tracker for Scenario Two.

Scenario Three is the multiple target scenario. The EKF-GNN and the CMKF-GNN are tested for this scenario, and the results are shown in Figures 6-7. Figures 6-7 show that sometimes the EKF performs better, while sometimes the CMKF performs better at initiating and maintaining tracks, but overall their performance is quite similar.

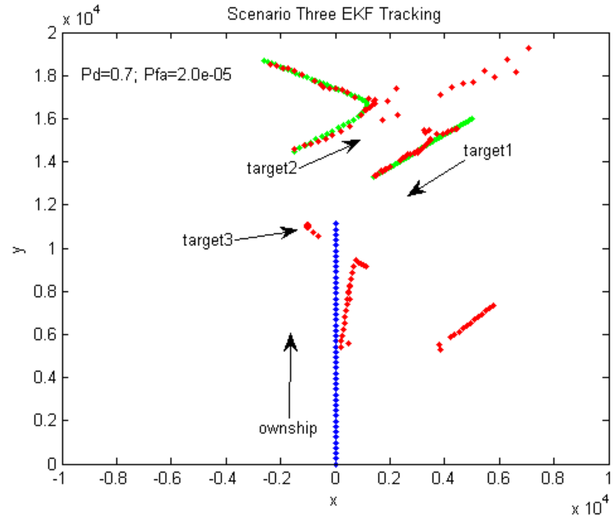

Figure 6. Tracks produced using the EKF-GNN tracker for Scenario Three.

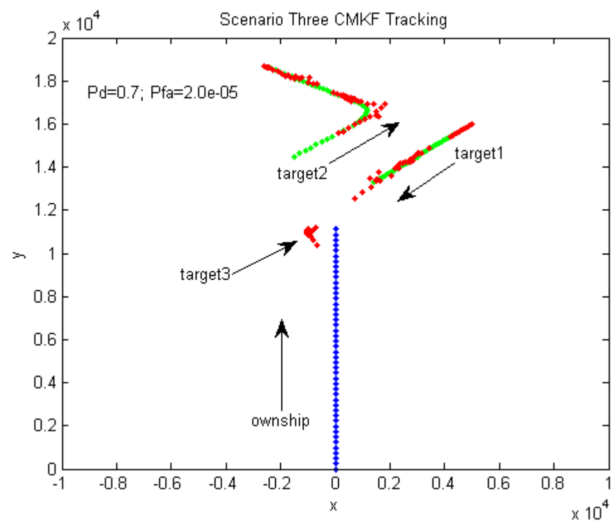

Figure 7. Tracks produced using the CMKF-GNN tracker for Scenario Three.

Tables 1 and 2 show the relative performance, using metrics defined in Section III.B, of these trackers for the above scenarios over 10 runs.

TABLE I
RESULTS OF EKF FOR SCENARIOS 1-3

Metric	EKF		
	Scenario One	Scenario Two	Scenario Three
RMS Error	1.20	0.57	1.20
NfalseTracks	2	4.4	2
Fragmentation	2.62	2.4	2.62
Latency	160 seconds	300 seconds	160 seconds
Track Pd	0.84	0.81	0.84

TABLE 2
RESULTS OF CMKF FOR SCENARIOS 1-3

Metric	CMKF		
	Scenario One	Scenario Two	Scenario Three
RMS Error	0.29	0.31	0.68
NfalseTracks	2.9	2.3	3.2
Fragmentation	1.6	1.8	2.17
Latency	120 seconds	200 seconds	120 seconds
Track Pd	0.92	0.85	0.84

For the three scenarios considered, both the EKF and CMKF give similar performance, with the CMKF producing slightly lower *rms* errors, and track fragmentation. For both trackers, latency increases for the manoeuvring target in Scenario Two.

V. SUMMARY

An active sonar tracking software suite for Anti-submarine Warfare applications has been developed in Matlab. It comprises an active sonar scenario generator, an active tracker and a capability assessment tool. This allows us to assess the performance achieved with different active sonar tracking algorithms and different sonar systems.

A study of the performance of active sonar trackers for three active sonar ASW scenarios was undertaken using simulated data. In the CMKF, the measurement covariance matrix, the innovation covariance matrix and filter gain are all dependent on the individual measurement at a given time, and so a new and previously unpublished CMKF-PDA was developed by deriving the expressions for the updated state estimate and the updated state estimate error covariance matrix for the CMKF. Several performance metrics were defined. The rules for automatic track initiation, track maintenance and track termination were described. The study showed that, for both single target and multiple targets scenarios, the EKF and CMKF had no significant differences in terms of tracker performance. The reason for this may be that the tracking systems for ASW applications are not highly nonlinear because of the relatively slow moving nature of the ocean vessels, unlike in some other aerospace applications.

ACKNOWLEDGMENT

The authors would like to thank Dr. Jane Thredgold for providing the active sonar scenario generation software.

REFERENCES

[1] L. Mo, X. Song, Y. Zhou, Z. Sun, and Y. Bar-Shalom, "Unbiased converted measurements for tracking", IEEE Transactions on aerospace and electronic systems, Vol. 34, No. 3, July 1998.
[2] Z. Duan, C. Han, and X. Rong Li, Comments on "Unbiased converted measurements for tracking", IEEE Transactions on aerospace and electronic systems, Vol. 40, No. 4, October 2004.
[3] F. Fletcher and D. Kershaw, "Performance analysis of unbiased and classical measurement conversion techniques", IEEE Transactions on aerospace and electronic systems, Vol. 38, No. 4, October 2002
[4] Y. Bar-Shalom, and T. Fortmann, "Tracking and data association", Academic Press, Inc. 1988.
[5] R. Karlsson and F. Gusafsson, "Monte Carlo Data Association for Multiple Target Tracking", IEE Target Tracking: Algorithms and Applications, 2001.
[6] P. Konstantinova, A. Udvarev and T. Semerdjiev, "A Study of a Target Tracking Algorithm Using Global Nearest Neighbour Approach", Proceedings of the 4th International Conference on Computer Systems and Technologies, 290-295, 2003.
[7] B. Ristic and N. Gordon, "Multisensor tracking and identification", Lecture notes, CSSIP, 2005.
[8] J. Thredgold, S. Lourey, H. Vu and M. Fewell, "A Simulation Model of Networked Tracking for Anti-Submarine Warfare", DSTO-TR-2372, 2010.
[9] S. Coraluppi, D. Grimmet and P. de Theije, "Benchmark Evaluation of Multistatic Trackers", FUSION 2006, The 9th International Conference on Information Fusion 2006, Florence (Italy), Conference Proceedings.

APPENDIX

The detailed steps for determining the updated state estimate $\hat{X}(k \mid k)$ and the error covariance matrix $P(k \mid k)$ for the updated state estimate are given below. The following notations are adopted, which are consistent with those used in [4].

$\hat{X}(k \mid k-1)$ is the predicted state estimate,

$\hat{X}(k \mid k)$ is the updated state estimate,

$P(k \mid k)$ is the error covariance matrix for the updated state estimate

$P(k \mid k-1)$ is the error covariance matrix for the filter predicted state estimate,

m_k is the number of measurements in the gate at time *k*,

$\theta_0(k)$ is the event that none of the measurements in the gate at time *k* originated from the target,

$\theta_i(k)$ is the event that that the *i*-th measurement ($1 \le i \le m_k$) in the gate at time *k* originated from the target,

$\beta_0(k)$ is the probability that $\theta_0(k)$ occurs,

$\beta_i(k)$ is the probability that $\theta_i(k)$ ($1 \le i \le m_k$) occurs , and β_0, β_i are evaluated as in [4].

$\nu_i(k)$ is the difference between the *i*-th measurement ($1 \le i \le m_k$) and the filter predicted measurement,

Z^k is the set all validated measurements, and

$W_i(k)$ is the filter gain for the *i*-th measurement ($1 \le i \le m_k$) at time *k*.

Using the total probability theorem, the predicted state vector can be written as

$$\hat{X}(k \mid k) = \sum_{i=0}^{m_k} \hat{X}_i(k \mid k)\beta_i(k)\,. \qquad (1)$$

Since

$$\hat{X}_i(k \mid k) = \hat{X}(k \mid k-1) + W_i(k)\nu_i(k)$$

for $i = 1, \ldots, m,$ and

$$\hat{X}_0(k \mid k) = \hat{X}(k \mid k-1)\,,$$

Equation (1) becomes

$$\hat{X}(k \mid k) = \hat{X}(k \mid k-1) + \sum_{i=1}^{m_k} \beta_i(k) W_i(k)\nu_i(k) \qquad (2)$$

which gives the expression for the updated state estimate.

The error covariance matrix for the updated state estimate

$$P(k \mid k) = E\{[X(k) - \hat{X}(k \mid k)][X(k) - \hat{X}(k \mid k)]^T \mid Z^k\}$$

$$= \sum_{i=0}^{m_k} \beta_i(k) E\{[X(k) - \hat{X}(k \mid k)][X(k) - \hat{X}(k \mid k)]^T \mid \theta_i(k), Z^k\}$$

$= A + B + B^T + C,$ where

$$A = \sum_{i=0}^{m_k} \beta_i(k) E\{X(k)X(k)^T) \mid \theta_i(k), Z^k\}$$

$$= \sum_{i=0}^{m_k} \beta_i(k)[\hat{X}_i(k \mid k)\hat{X}_i(k \mid k)^T + P_i(k \mid k)]$$

with

$P_0(k \mid k) = P(k \mid k-1)$ and

$P_i(k \mid k) = [I - W_i(k)H(k)]P(k \mid k-1)$ for $i = 1, \ldots, m_k$.

Thus

$$A = \beta_0(k)P(k \mid k-1) + \sum_{i=1}^{m_k} \beta_i(k)[I - W_i(k)H(k)]P(k \mid k-1)$$

$$+ \sum_{i=0}^{m_k} \beta_i(k)\hat{X}_i(k \mid k)\hat{X}_i(k \mid k)^T.$$

Also,

$B = -\hat{X}(k \mid k)\hat{X}(k \mid k)^T = B^T$ and

$C = -B.$

Therefore,

$$P(k \mid k) = A + B$$

$$= \beta_0(k)P(k \mid k-1) + \sum_{i=1}^{m_k} \beta_i(k)[I - W_i(k)H(k)]P(k \mid k-1) + \widetilde{P},$$

where

$$\widetilde{P} = \sum_{i=0}^{m_k} \beta_i(k)\hat{X}_i(k \mid k)\hat{X}_i(k \mid k)^T - \hat{X}(k \mid k)\hat{X}(k \mid k)^T.$$

Now, using (1) and (2), $\widetilde{P}$ can be simplified as

$$\widetilde{P} = \sum_{i=0}^{m_k} \beta_i(k)\{\hat{X}(k \mid k-1) + W_i(k)\nu_i(k)\}\{\hat{X}(k \mid k-1) + W_i(k)\nu_i(k)\}^T -$$

$$\{\hat{X}(k \mid k-1) + \sum_{i=0}^{m_k} \beta_i(k)W_i(k)\nu_i(k)\} \times$$

$$\{\hat{X}(k \mid k-1) + \sum_{i=0}^{m_k} \beta_i(k)W_i(k)\nu_i(k)\}^T$$

$$= \sum_{i=1}^{m_k} \beta_i(k)W_i(k)\nu_i(k)\nu_i(k)^T W_i(k)^T - \sum_{i=1}^{m_k} \beta_i(k)W_i(k)\nu_i(k) \sum_{i=1}^{m_k} \beta_i(k)\nu_i(k)^T W_i(k)^T,$$

because $\sum_{i=0}^{m_k} \beta_i(k) = 1$ and $\nu_0(k) = 0$ for all k.

Finally, the expression for $P(k \mid k)$ is obtained

$$P(k \mid k) = \beta_0(k)P(k \mid k-1) + \sum_{i=1}^{m_k} \beta_i(k)[I - W_i(k)H(k)]P(k \mid k-1)$$

$$+ \sum_{i=1}^{m_k} \beta_i(k)W_i(k)\nu_i(k)\nu_i(k)^T W_i(k)^T -$$

$$\sum_{i=1}^{m_k} \beta_i(k)W_i(k)\nu_i(k) \sum_{i=1}^{m_k} \beta_i(k)\nu_i(k)^T W_i(k)^T\,.$$