Detection of Weak Signals in Non-Gaussian Noise Using Nonlinear Wavelet Denoising

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***Abstract*- In this paper, we present a nonlinear suboptimal detector whose performance in the leptokurtic noise is significantly better than that of matched filter. This detector contains a denoising pre-processor based on a non-linear wavelet transform called median interpolating pyramid transform (MIPT) proposed by Donoho and Yu [1]. The output of the pre-processor is passed through a matched filter and a threshold detector to decide whether the signal is present. The performance analysis of this nonlinear wavelet denoising (NWD) detector has been done by deriving the relation between the PDF of denoised signal and the PDF of input signal. Thus the mean and the variance of the test statistic are determined to provide the receiver operating characteristic (ROC) of the NWD detector in the asymptotic case. The theoretical analysis, confirmed by the experimental results, shows a significant improvement in the performance of the proposed detector over that of the matched filter in non-Gaussian noise.**

I. INTRODUCTION

Detection of weak acoustic signals is a problem of interest in several underwater applications related to the ocean. It is known that ambient noise in the ocean is leptokurtic (heavy-tailed) in many situations [2]. When noise is leptokurtic, the matched filter which provides the optimal detection performance in Gaussian noise is not the optimal choice. On the other hand, implementation and performance analysis of the optimal (in the Neyman-Pearson sense) detector in non-Gaussian noise is a difficult task. Hence suboptimal detectors that are more convenient to implement are usually used for detection of signals in non-Gaussian noise.

In this paper, we present a nonlinear suboptimal detector based on non-linear wavelet denoising (NWD) whose performance in non-Gaussian noise is considerably better than that of the matched filter (MF). Moreover, we analyse the performance of this detector theoretically and verify the theoretical analysis with experimental results. The performance analysis is done by deriving the relation between the probability density function (PDF) of denoised data and the PDF of received data under hypothesis H_1 (signal present) and hypothesis H_0 (signal absent). Thus the mean and the variance of the test statistic are determined under both hypotheses. These values of mean and variance are used to determine the receiver operating characteristic (ROC) of the NWD detector in the asymptotic (N -> infinity) case. This paper is organized as follows. In Section II, the nonlinear wavelet denoising technique used in the NWD-detector is explained. The structure of the proposed detector is described in Section III. Section IV includes the related details of theoretical performance analysis of the proposed detector. Results and conclusion are presented in Section V and Section VI respectively.

II. NONLINEAR WAVELET DENOISING BASED ON MEDIAN INTERPOLATION

In general, wavelet denoising attempts to recover the signal $s(n);\ n = 0, 1, 2, \ldots, N-1$ from the noise-corrupted observations $x(n)=s(n)+w(n)$ where $w(n)$ are noise samples. Wavelet transform compacts the signal into a few large coefficients and the process of wavelet denoising is to threshold the coefficients to discard the values most likely due to the additive noise. In the proposed detector, the denoising or SNR enhancement is achieved using a non-linear wavelet transform called the median interpolating pyramid transform (MIPT) proposed by Donoho and Yu [1]. The core of this method is the median interpolating (MI) refinement scheme. A block diagram of the nonlinear wavelet denoising method is presented in Fig. 1.

In this method, MIPT replaces the conventional linear wavelet transform employed for denoising signals in Gaussian noise. The dimension $N=3^J$ of the data vector is chosen to be an integer power of 3. To determine the MIPT coefficients at scale j: $0< j <J$, the data set is divided into 3^j blocks of size 3^{J-j} and the medians $m_{j,k}$ $(k=0,1,\ldots,3^j-1)$ of the blocks are computed. Medians $m_{j+1,3k+l}$ $(l=0,1,2)$ at the finer scale $j+1$ are then estimated (imputed) from the median $m_{j,k+r}$ $(r= -1,0,1)$ at the coarser scale j using a quadratic interpolating polynomial chosen such that it passes through the medians at scale j.

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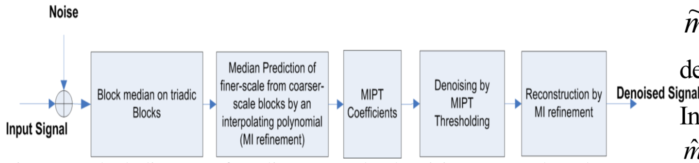

Figure 1: Block diagram of nonlinear wavelet denoising (NWD) based on median interpolation

The estimated medians denoted by $\hat{m}_{j+1,k}$ $(k = 0,1,\ldots,3^{j+1}-1)$ are called imputed medians. The differences

$$\varepsilon_{j,k} = m_{j,k} - \hat{m}_{j,k} \;\; (k = 0,1,\ldots 3^{j}-1; \;\; j = J, J-1,\ldots, j_0+1) \quad (1)$$

are the MIPT coefficients for a j_0-level wavelet decomposition for $j_0 \le J$. The unique relationship between $\hat{m}_{j+1,3k+l}$ $(l = 0,1,2)$ and $m_{j,k+r}$ $(r = -1,0,1)$ is nonlinear. The original time series can be reconstructed from the MIPT coefficients by reversing the above mentioned procedure. Denoising is achieved by subjecting the MIPT coefficients $\varepsilon_{j,k}$ to hard thresholding. The thresholds are level-dependent, and are also dependent on PDF of noise. It has been shown by Donoho and Yu [1] that for zero-mean IID (independent and identically distributed) noise with symmetric PDF, the magnitude $\left|\varepsilon_{j,k}\right|$ of every MIPT coefficient is very likely to be less than the threshold t_j defined as:

$$t_j = F^{-1}\left(\frac{1}{2} + \frac{1}{2}\sqrt{1-\left(\frac{1}{2J3^{J}}\right)^{\frac{2}{3^{J-j}}}}\right), \quad (2)$$

where F(.) denotes the CDF (Cumulative distribution function) of noise. The MIPT coefficients with absolute value less than t_j may be considered as noise and neglected.

MIPT coefficients of the denoised signal are given by:

$$\tilde{\varepsilon}_{j,k} = \begin{cases} \varepsilon_{j,k} & if \; \left|\varepsilon_{j,k}\right| \; \ge \; t_j \\ 0 & if \; \left|\varepsilon_{j,k}\right| \; < t_j \end{cases}$$

$$j = j_0+1, j_0+2, \ldots, J \quad (3)$$

The medians $\tilde{m}_{j,k}$ of the denoised signal are reconstructed using the equations:

$$\tilde{m}_{j_0,k} = m_{j_0,k}$$

$$\tilde{m}_{j\;,k} = \tilde{\varepsilon}_{j,k} + \hat{\tilde{m}}_{j,k} \;\; , j = j_0+1,\ldots, J \quad (4)$$

where $\hat{\tilde{m}}_{j,k}$ is the imputed denoised median at scale *j*. Once again, $\hat{\tilde{m}}_{j+1,3k+l}$ $(l = 0,1,2)$ at scale j+1 are estimated from $\tilde{m}_{j\;,k+r}$ $(r = -1,0,1)$ at scale j using the interpolation procedure described above.

In linearized MIPT (LMIPT), the relation between $\hat{m}_{j+1,3k+l}$ $(l = 0,1,2)$ and $m_{j,k+r}$ $(r = -1,0,1)$ is linearized. It can be shown that error due to linearization is small. LMIPT is used for denoising the received signal in the NWD detector proposed in this paper.

III. NWD DETECTOR

We consider the hypothesis testing problem:

H_0: x(n)=w(n)
H_1: x(n)=w(n)+s(n), n=0, 1, .. ,N-1

Under hypothesis H_1, the data vector $\boldsymbol{x}$ consists of a known deterministic signal $\boldsymbol{s}$ contaminated by additive random noise $\boldsymbol{w}$. Under hypothesis H_0, $\boldsymbol{x} = \boldsymbol{w}$.

The test statistic of the conventional detector (matched filter) shown in Fig. 2(a) is given by:

$$T(\mathbf{x}) = \sum_{n=0}^{N-1} x(n)\, s(n) \cdot$$

In channels with Gaussian noise, the matched filter (MF) is optimal. But it is not optimal in non-Gaussian noise. Though it is possible to determine the optimal (in the Neyman-pearson sense) detector in non-Gaussian noise, the implementation of the optimal detector is not easy, in general. In this paper, we propose an easily implementable nonlinear sub-optimal detector based on NWD, and show that its performance is much better than that of MF. The block diagram of the NWD detector is shown in Fig. 2(b).

In the proposed detector, the denoising transformation based on LMIPT is applied to the data vector $\boldsymbol{x}$ to obtain the transformed vector $\boldsymbol{y}$. The denoising pre-processor enhances the signal-to-noise ratio (SNR) of the noisy signal under hypothesis H_1. The output of the pre-processor is passed through a matched filter and a threshold detector to decide whether the signal is present.

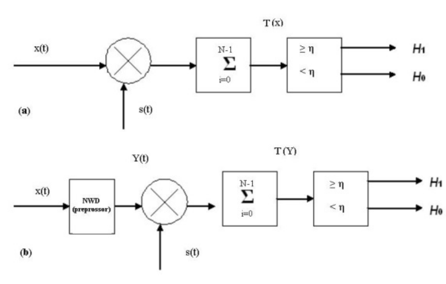

Figure 2 : a) Matched filter detector, b) NWD detector

The test statistic of the proposed detector is given by:

$$\mathrm{T(y)} = \sum_{n=0}^{N-1} y(n)\, s(n)\,, \qquad (5)$$

where y=$[y(0)...y(N-1)]^T$ is the vector obtained through the NWD transformation of data vector $\boldsymbol{x}$. The objective of this transformation is to enhance the SNR if the data vector contains the signal component.

We compare the performance of the NWD detector with that of MF and LOD [3] (locally optimal detector). We assume that the noise *w(n)* is heavy-tailed (leptokurtic) with generalized Gaussian PDF given by:

$$f_w(x) = a \exp\{-b|x|^p\} \quad ,0 < p < 2$$

where

$$a = \frac{p}{2} \times \frac{\left[\Gamma(\frac{3}{p})\right]^{1/2}}{\left[\Gamma(\frac{1}{p})\right]^{3/2}}, \quad b = \left[\frac{\Gamma(\frac{3}{p})}{\Gamma(\frac{1}{p})}\right]^{p/2} \quad and\ \Gamma(\alpha) = \int_0^{\infty} \exp(-t)\, t^{\alpha-1}\, dt$$

IV. The Performance Analysis of NWD-detector

The first step in the performance analysis of NWD-detector is to obtain the PDF of *y(n)* under the hypotheses H1 H0. This involves the computation of PDF of the signal at the output of each block in Fig. 1. The output PDF of each block is considered as the input PDF of next block, and finally the PDF of NWD transformed signal *y(n)* is computed in terms of PDF of received data *x(n)*. As mentioned in SectionI, the median interpolation refinement scheme proposed by Donoho and Yu [1] is a nonlinear scheme. By linearized approximation of this refinement method (which has been examined by us to be a well-approximated version of exact MIPT) the relationship between the imputed medians at scale *j+1* is determined as a linear combination of the exact medians at scale *j* :

$$\hat{m}_{j+1}(3k+l) = imp(m_{j,k-1}, m_{j,k}, m_{j,k+1}) = \alpha_{-1}(l) m_{j,k-1} + \alpha_0(l) m_{j,k} + \alpha_1(l) m_{j,k+1} \qquad l = 0,1,2 \qquad (6)$$

where

$$\begin{bmatrix} \alpha_{-1}(0) & \alpha_0(0) & \alpha_1(0) \\ \alpha_{-1}(1) & \alpha_0(1) & \alpha_1(1) \\ \alpha_{-1}(2) & \alpha_0(2) & \alpha_1(2) \end{bmatrix} = \begin{bmatrix} 2/9 & 8/9 & -1/9 \\ 0 & 1 & 0 \\ -1/9 & 8/9 & 2/9 \end{bmatrix} \qquad (7)$$

and 'imp' is the Median-Interpolating refinement operator .
Using the definition of the MIPT coefficients in (1), we have:

$$\begin{aligned} \varepsilon_{j+1,3k+l} &= m_{j+1,3k+l} - \hat{m}_{j+1,3k+l} \\ &= m_{j+1,3k+l} - \alpha_{-1}(l).m_{j,k-1} - \alpha_0(l).m_{j,k} - \alpha_1(l).m_{j,k+1} \\ &= z(k,l) + w(k,l) + v(k,l) \end{aligned} \qquad (8)$$

where

$$\begin{aligned} z(k,l) &= m_{j+1,3k+l} - \alpha_0(l)\, m_{j,k} \\ w(k,l) &= -\alpha_{-1}(l)\, m_{j,k-1} \\ v(k,l) &= -\alpha_1(l)\, m_{j,k+1} \quad ,l = 0,1,2 \end{aligned} \qquad (9)$$

The random variables *Z(k,l)*, *W(k,l)* and *V(k,l)* are mutually independent for all *k* due to the non-overlapping nature of the pyramidal transform and the assumption that the data samples *x(n)* are mutually independent random variables.

It is seen from (8) and (9) that for obtaining the PDF of the MIPT coefficients, we have to determine the PDF of $m_{j,k}$, and the joint PDF of $m_{j,k}$ and $m_{j+1,3k+l}$ *(l=0,1,2)*. It can be shown that if *x(n)* are IID random variables with PDF *f(x)* and CDF *F(x)*, the PDF of $m_{j,k}$, denoted by $f_j(x)$, and the joint CDF of $m_{j,k}$ and and $m_{j+1,3k+l}$ *(l=0,1,2)* denoted by $F_{j,j+1}(x,y)$ are given by [4]:

$$f_j(x) = (6u+3)\binom{6u+2}{3u+1} F^u(x)[1-F(x)]^u\, f(x), \qquad (10)$$

and

$$F_{j\,j+1}(x,y) = P(m_{j,k} \le x, m_{j+1,3k+l} \le y) \qquad (11a)$$

$$= \sum_{i=u+1}^{2u+1} \sum_{t=0}^{2u+1-i} \sum_{m=0}^{u+i+t} \{\binom{4u+2}{3u+2+m-(i+t)} \frac{(2u+1)!}{i!\,t!\,(2u+1-i-t)!} \times \{F^i(y)\ F^{3u+2+m-i-t}(x)\ (F(x)-F(y))^t\ (1-F(x))^{1+3u-m}\}\} \quad : x > y$$

$$F_{j\,j+1}(x,y) = \sum_{i=u+1}^{2u+1} \sum_{t=0}^{i} \sum_{m=0}^{u+t} \{\binom{4u+2}{3u+2+m-t} \frac{(2u+1)!}{t!\,(i-t)!\,(2u+1-i)!} \times \{F^{3u+2+m}(x)\ \ (F(y)-F(x))^{i-t}\,(1-F(y))^{2u+1-i}\,(1-F(x))^{u-m+t}\}\} \quad : x < y \qquad (11b)$$

$$F_{j\,j+1}(x,y) = \sum_{i=u+1}^{2u+1} \sum_{m=0}^{u+i} \binom{4u+2}{3u+2+m-i} \frac{(2u+1)!}{i!\ \ (2u+1-i)!}\ F^{3u+2+m}(x)\ \ (1-F(x))^{1+3u-m} \qquad (11c)$$

where $u = \frac{1}{2}(3^{J-j-1} - 1)$.

The joint PDF can be found by differentiating the joint CDF with respect to *x* and *y*.

The PDF of random variables *V(k,l)*, *W(k,l)* and *Z(k,l)* can be expressed as:

$$f_w(t;l) = \frac{1}{|\alpha_{-1}(l)|} f_j(\frac{-t}{\alpha_{-1}(l)}), \qquad (12)$$

$$f_v(t;l) = \frac{1}{|\alpha_1(l)|} f_j(\frac{-t}{\alpha_1(l)}), \qquad (13)$$

$$f_z(t;l) = \int_{-\infty}^{+\infty} \frac{1}{|\alpha_0(l)|} f_{j,j+1}(\frac{-v}{\alpha_0(l)}, t-v)\ dv\,, \qquad (14)$$

Finally the PDF of the MIPT coefficients at scale j+1 can be obtained by computing the convolution of PDF of *Z(k,l)*, *W(k,l)* and *V(k,l)*. So, we have:

$$f_{\varepsilon_{j+1,3k+l}}(t;l) = f_z(t;l) * f_w(t;l) * f_v(t;l) \tag{15}$$

where * denotes convolution.

The PDF of the thresholded MIPT coefficient $\tilde{\varepsilon}_{j,k}$ is given by:

$$\begin{aligned} f_{\tilde{\varepsilon}_{j,k}}(t) &= \left[F_{\varepsilon_{j,k}}(t_j) - F_{\varepsilon_{j,k}}(-t_j)\right]\delta(t) \\ &\quad + f_{\varepsilon_{j,k}}(t)\left[U(t-t_j) + U(-t-t_j)\right], \\ &\quad j = j_0+1, j_0+2, \ldots, J \end{aligned} \tag{16}$$

where t_j is the threshold defined in (2), and δ(t) and U(t) are the Dirac delta function and unit step function respectively. We note that (i) $\tilde{m}_{j_0,k} = m_{j_0,k}$ and $\tilde{m}_{j,k} = \tilde{\varepsilon}_{j,k} + \hat{\tilde{m}}_{j,k}$ for $j_0 < j \le J$ (see Eq. (4)), (ii) the relation between the imputed denoised median $\hat{\tilde{m}}_{j+1,3k+l}\ (l = 0,1,2)$ and the denoised medians $\tilde{m}_{j_0,k+r}\ (r = -1,0,1)$ is governed by the transition matrix defined in (7), and (iii) the NWD-transformed signal is given by $y(k) = \tilde{m}_{J,k}$, $k = 0,1,\ldots,3^J - 1$.

Hence, the PDFs of $\tilde{m}_{j,k}$ can be determined recursively starting from $j=j_0$, and eventually the PDF of *y(n)* can be obtained. Details of this recursive procedure are given in [4].

The procedure outlined above can be used to determine the PDF of *y(n)* under both hypotheses if the noise *w(n)* is IID and the signal s(n) is a constant . Under these conditions, and assuming GG noise, the PDF of the data *x(n)* is given by:

Hypothesis H0:

$$f_x(x) = a \exp\{-b|x|^p\} \tag{17}$$

Hypothesis H1:

$$f_x(x) = a \exp\{-b|x-s|^p\} \quad , p > 0 \tag{18}$$

By using the PDFs in (17) and (18) as input PDFs, the PDF of *y(n)* under each hypothesis can be computed numerically. After that, we can compute the mean and variance of *y(n)* under both hypotheses. Using these values of mean and variance, we can determine probability of false alarm P_{FA}, probability of detection P_D and receiver operating characteristic (ROC) of the NWD detector in the asymptotic (N -> infinity) case [3] by invoking the central limit theorem.

$$\begin{aligned} P_{FA} &= pr\{T(y) > \eta; H_0\} \\ &= Q\left(\frac{\eta - E(T;H_0)}{std(T;H_0)}\right) \\ P_D &= pr\{T(y) > \eta; H_1\} \\ &= Q\left(\frac{\eta - E(T;H_1)}{std(T;H_1)}\right) \\ &= Q\left(r Q^{-1}(P_{FA}) - \sqrt{r d^2}\right) \end{aligned} \tag{19}$$

where

$$d^2 = \frac{(E(T;H_1) - E(T;H_0))^2}{std(T;H_0) \times std(T;H_1)}, \quad r = \frac{std(T;H_0)}{std(T;H_1)}, \tag{20}$$

Q(x) is the complementary CDF of a standard normal random variable, Q^{-1} is the inverse of the *Q* function, *E(T)* is the mean of *T*, and *std(T)* is the standard deviation of *T*. It can be readily shown that, for a constant signal ***s***,

$$\begin{aligned} E(T;H_i) &= N\, s\, E(y(n);H_i)\,; & i = 0,1 \\ \mathrm{var}(T;H_i) &= N\, s^2\, \mathrm{var}(y(n);H_i); & i = 0,1 \end{aligned} \tag{21}$$

where *var* denotes variance.

V. Results and Discussion

The theoretical and experimental (simulation) results are presented in this section. In our simulations, we considered a data vector of 3^5 samples, a constant signal, and IID samples of leptokurtic GG noise with different values of *p*. The input ENR (signal energy-to-noise power ratio) and SNR in our experiments is -2.16dB and -26.02dB respectively. In Fig. 3, the PDF of input signal (***x***) is shown under both hypotheses for GGN with *p=0.55*. The PDF of the NWD-transformed signal (***y***) for these two input PDFs are illustrated in Fig. 4. These output PDFs were computed using formulas in Section IV for NWD transformed data with 2 levels of decomposition. The close up of the peaks in the output PDF are shown in Fig. 4(b) and 4(c).

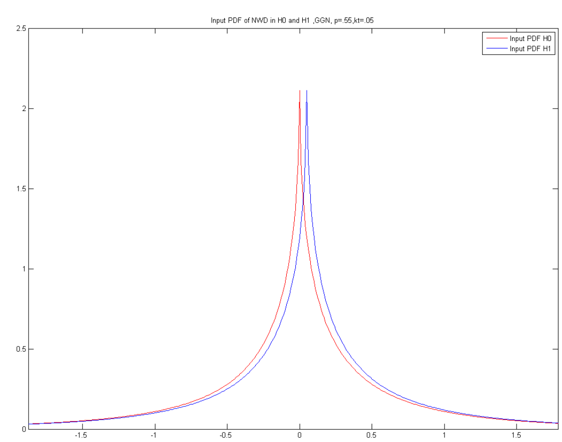

Figure 3: PDF of input (received) signal under hypotheses H0, H1 in GG noise, p=0.55

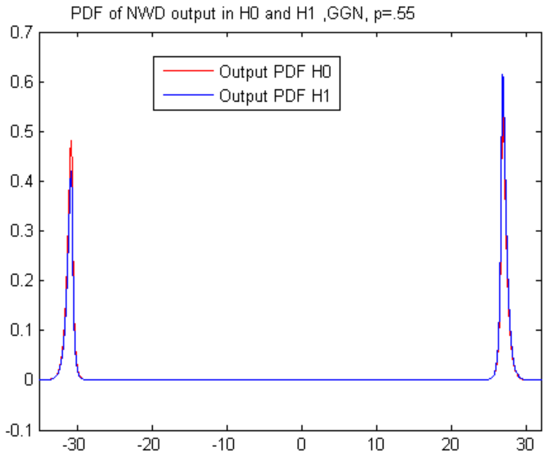

Fig. 4(a): PDF of y(n) (NWD output) under hypotheses H0, H1 when the input PDF is as in Fig. 3

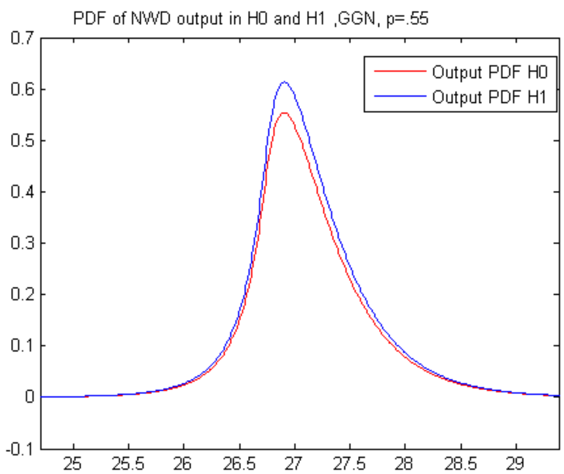

Fig. 4(b): Close-up of right-side peaks in Fig. 4(a)

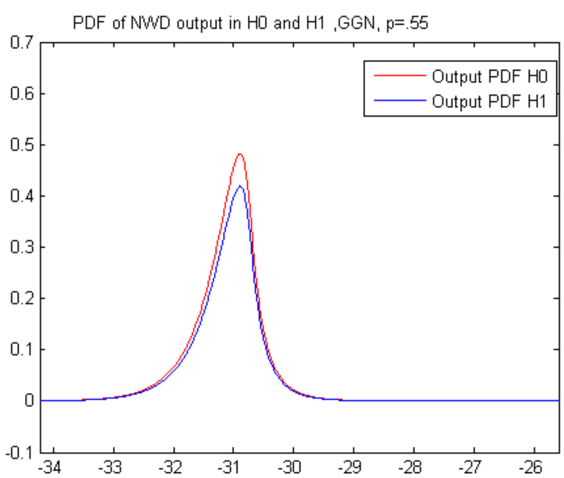

Fig. 4(c): Close-up of left-side peaks in Fig. 4(a)

In Fig. 5 receiver operating characteristic (ROC) of NWD-detector is compared with that of MF and LOD experimentally and theoretically for GGN with $p=0.55$. The data length is 3^6 samples. The experimental results were obtained using 20000 Monte Carlo simulations. It is seen that there is an excellent match between the theoretical and experimental ROC plots for the NWD and MF detectors. The performance of NWD detector is significantly better than that of MF. The experimental ROC plots of LOD and NWD detectors are quite close to one another.

In Fig. 6, the data length is 3^5 samples. All the other parameters have the same values as in Fig. 5. This figure shows degradation in the performance of all detectors. Still, the NWD detector performs much better than MF.

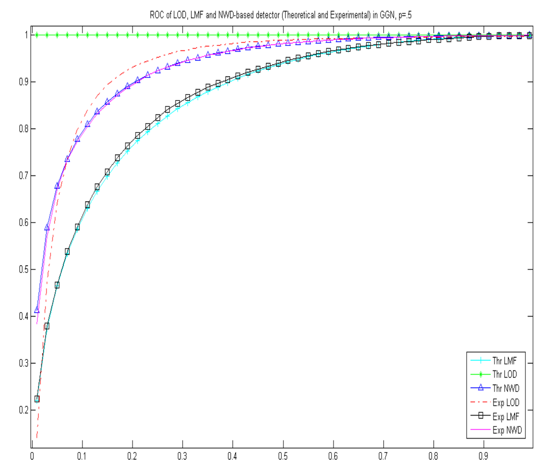

Figure 5: ROC of MF and NWD-detector for GGN, p=0.55, Number of samples=3^6 (Black line with square: Experimental MF, Cyan line with plus: Theoretical MF, Magenta solid line: Experimental NWD-detector, Blue line with triangle: Theoretical NWD-detector, Green line with star: Theoretical LOD, Red dash-dot line: Experimental LOD)

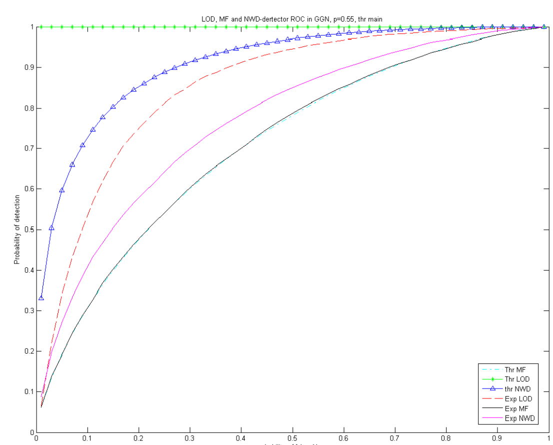

Figure 6: ROC of MF and NWD-detector for GGN, p=0.55, Number of samples=3^5 (Black solid line: Experimental MF, Cyan dash-dot line: Theoretical MF, Magenta solid line: Experimental NWD-detector, Blue line with triangle: Theoretical NWD-detector, Green line with star: Theoretical LOD, Red dash line: Experimental LOD)

In another experiment, we have examined the effect of increasing the number of levels of decomposition on the performance of NWD detector. A comparison of the ROCs of NWD detector for 2 and 3 levels of decomposition is presented in Fig. 7. It is observed that increasing the number of decomposition levels from 2 to 3 leads to a significant improvement in performance. The price paid for this improvement is greater computational complexity.

In all the results presented so far, we have assumed that the noise is GG with $p = 0.55$. The effect of variation of the parameter p on the performance of NWD detector is illustrated in Fig. 8. This figure shows the variation of deflection coefficient d^2 (defined in (20)) as the parameter p is varied, for MF, LOD, and NWD detector with 2 levels of decomposition. For MF, which does not involve any preprocessing of the data, d^2 is independent of the noise PDF. For NWD, d^2 varies with p. However, d^2 for the NWD detector is always higher than d^2 for MF, which means that the performance of the NWD detector would be better than that of MF for the entire class of leptokurtic GG noise distributions. For LOD, d^2 increases monotonically as p is reduced, and hence the performance of LOD keeps on improving as noise becomes more leptokurtic.

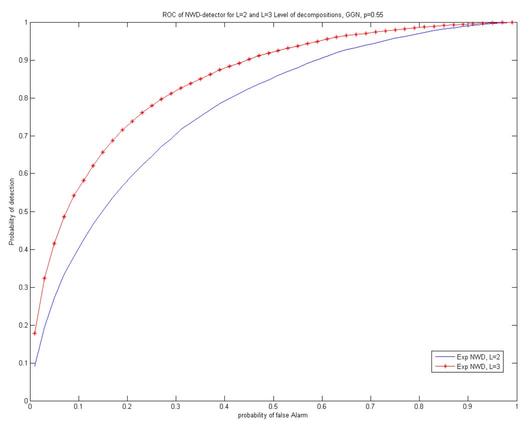

Figure 7: Experimental ROCs of NWD detector with L = 2 and 3 levels of decomposition for GGN, p = 0.55, ENR = -2.16 dB

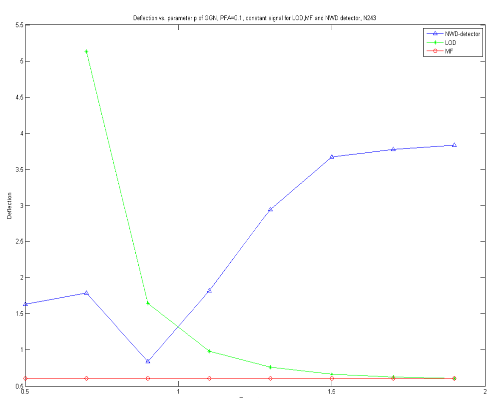

Figure 8: Variation of deflection coefficient d^2 with parameter p of GG noise

VI. CONCLUSION

In this paper, we have presented an easily implementable detector based on nonlinear wavelet denoising for detection of signals in leptokurtic noise. In the proposed detector, a denoising transformation based on linearized MIPT is applied to the data vector and the transformed data vector is correlated with the signal vector to obtain the test statistic. The theoretical analysis of the detector has been done by obtaining the PDF of the transformed data in terms of the PDF of the input data under the assumption of IID data samples. This assumption implies IID noise samples and a constant signal. The PDF of the transformed data vector has been used to determine the mean and variance of the test statistic under both hypotheses, and thus determine the asymptotic (N --> infinity) performance of the NWD detector. Receiver operating characteristics of the NWD detector in generalized Gaussian noise have been obtained theoretically and experimentally and have been shown to agree with one another. Performance of the NWD detector is significantly better than that of the matched filter. It has also been shown that the performance of the NWD detector can be improved by increasing the number of levels of MIPT decomposition from 2 to 3. But the issue of optimal choice of the number of decomposition levels has not been explored. Extension of the performance analysis of the NWD detector to the case of non-constant signals is currently in progress.

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