Notes on Deterministic Wave Elevation Prediction

Umesh A. Korde, Johns Hopkins University.

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1 Using Wave Crest Locations

Given an irregular long-crested wave field represented as,

$$\begin{aligned} \eta(x,t) &= \sum_{n=1}^{N} \left[A_n \cos(k_n x - \omega_n t) + B_n \sin(k_n x - \omega_n t)\right], \\ &= \sum_{n=1}^{N} \left[C_n \cos(k_n x - \omega_n t - \theta_n)\right];\ \tan\theta_n = \frac{B_n}{A_n}. \end{aligned} \tag{1}$$

At each crest-line, $x = x_m$, $m = 1, \ldots, M$,

$$\frac{\partial \eta}{\partial x} = 0; \tag{2}$$

At each crest $x = x_m, m = 1, \ldots, M$, at a time instant $t = t_o$,

$$\frac{\partial \eta}{\partial x} = \sum_{n=1}^{N} \left[-k_n A_n \sin(k_n x_m - \omega_n t_o) + k_n B_n \cos(k_n x_m - \omega_n t_o)\right]. \tag{3}$$

For each component n in the summation in equation (1),

$$r_n = \frac{B_n}{A_n} = \frac{\sin(k_n x_m - \omega_n t_o)}{\cos(k_n x_n - \omega_n t_o)}. \tag{4}$$

Averaging over M crests, for the nth frequency component,

$$\overline{r}_n = \frac{1}{M} \sum_{m=1}^{M} \frac{\cos(k_n x_m - \omega_n t_o)}{\sin(k_n x_n - \omega_n t_o)}. \tag{5}$$

When crest-lines are available at multiple time instants L, an additional average over L time instants will further improve accuracy. The phase angle θ_n associated with each frequency component within the snap-shot can be estimated using,

$$\theta_n = \tan^{-1}(\overline{r}_n). \tag{6}$$

When a nearby wave spectrum is available, $S(\omega)$ is approximately known. Then the amplitude C_n can be expressed as,

$$C_n = \sqrt{2S(\omega_n)\Delta\omega}. \tag{7}$$

A snap-shot of wave elevation based on the detected crest-lines can then be reconstructed as,

$$\eta(x,t) = \sum_{n=1}^{N} C_n \cos(k_n x - \omega_n t - \theta_n). \tag{8}$$

2 Fixed Time Method: snap-shots

In deep water, for uni-directional wave propagation, a kinematic model relating the wave elevation at point x_A to that at point x_B in the frequency domain can be expressed as,

$$\eta(x_B; i\omega) = \mathrm{e}^{-ik(\omega)d}\eta(x_A; i\omega). \tag{9}$$

where $k(\omega)$ using the deep-water dispersion relation is

$$k(\omega) = \frac{|\omega|\omega}{g}. \tag{10}$$

$k(\omega)$ has the same sign as ω. For most realistic surface wave spectra over which a wave energy device operates, ω is contained within finite approximate limits ω_l and ω_h. Because surface waves are dispersive, an impulsive excitation of the wave surface propagates over a range of group velocities $[v_{gmn}, v_{gmx}]$, where, for deep water,

$$\begin{aligned} v_{gmn} &= \frac{1}{2}\left(\frac{g}{\omega_{mx}}\right) \\ v_{gmx} &= \frac{1}{2}\left(\frac{g}{\omega_{mn}}\right) \end{aligned} \tag{11}$$

It is assumed here that wave activity very far from the device has no influence over the spatial scales of interest to the measurement and prediction procedure (so that it can be assumed to be negligible). A convenient initial illustration of the fixed-time approach is provided by the case of uni-directional propagation, where the prediction based on this approach must match that based on the reconstructed fixed-point measurement above.

For the fixed-time approach, a 'snap-shot' wave surface elevation is obtained over a spatial domain at time t, and prediction is needed at time $t+p$ at the location of the device. The appropriate impulse response function in this case, for uni-directional propagation is,

$$h_p(d;p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-ikd}\mathrm{e}^{i\sqrt{gk}p}dk \tag{12}$$

The propagation impulse response function can be shown to be,

$$\begin{aligned} h_p(d;p) =& \sqrt{\frac{\pi g}{2d}}\frac{p}{d}\left[\cos\left(\frac{gp^2}{4d}\right)\mathrm{C}\left(\sqrt{\frac{g}{d}}\frac{p}{2}\right)\right] \\ &+ \sqrt{\frac{\pi g}{2d}}\frac{p}{d}\left[\sin\left(\frac{gp^2}{4d}\right)\mathrm{S}\left(\sqrt{\frac{g}{d}}\frac{p}{2}\right)\right] \\ &+ \sqrt{\frac{\pi g}{2d}}\frac{p}{d}\left[\cos\left(\frac{gp^2}{4d}\right) + \sin\left(\frac{gp^2}{4d}\right)\right] \end{aligned} \tag{13}$$

This impulse response function provides a prediction p seconds ahead in time and at the location of the device centroid x_B, using a 'snap-shot' wave-elevation measurement at time t over a spatial distance D, where

$$\begin{aligned} D &= v_{gmx}p - v_{gmn}p \\ x_h &= x_o + D \\ x_B &= x_h + v_{gmn}p \end{aligned} \tag{14}$$

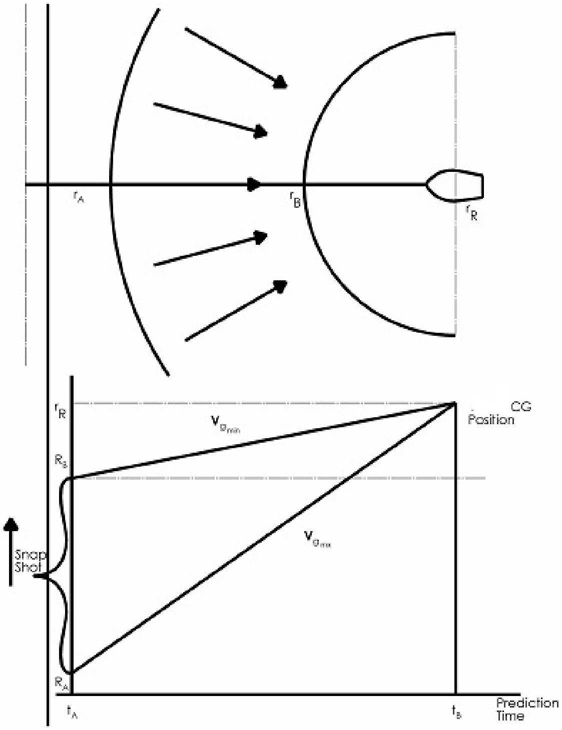

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Figure 1: A space-time diagram for the multi-directional prediction using a snap-shot of wave elevation measurements. Note that distances are now on the vertical axis, with time on the horizontal axis.

Then,

$$\eta(x_B; t+p) = \int_{x_o}^{x_h} h_p(x-\xi)\eta(\xi; t+p)d\xi \tag{15}$$

Note that, in this case, the device centroid is at x_B, and the snap-shot measurement stretches from x_o to $x_o + D$. x_o may be chosen based on the instrument measurement range providing the snap-shot measurement. A simulation result comparing the prediction obtained using equation (15) with the directly evaluated value at $(x_B; t+p)$ for an irregular wave record is shown in Figure 2.

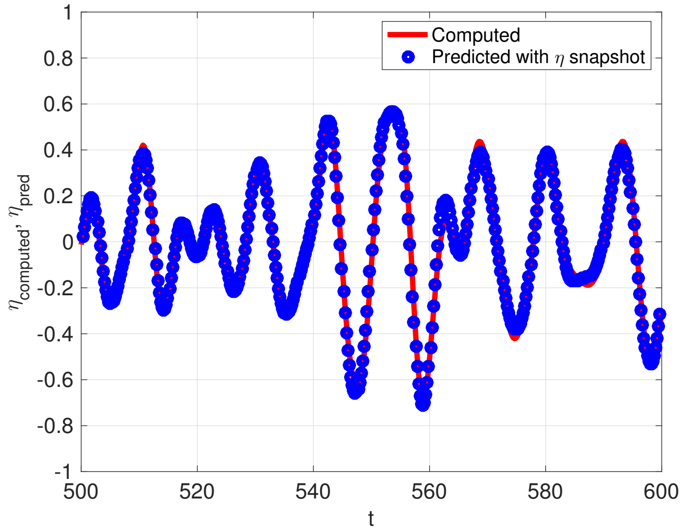

Figure 2: A comparison between the predicted wave elevation at the device centroid using a fixed-time measurement approach. The length of the snap-shot is 800m. The prediction time is 30s into the future. A long-crested wave field is assumed.